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INTERMEDIATE
TRIGONOMETRY

By

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CONTENTS

PUNJAB UNIVERSITY SYLLABUS

Trigonometry

For 1948 and after

Sexagesimal and circular units of angular measurement : trigonometrical ratios and the simple relations connecting them ; relations between trigonometrical ratios of angles differing by multiples of right angles; addition and substraction formulae, Logarithms ; solution of triangles and simple cases of heights and distances ; area of a circle ; graphs of simple trigonometrical functions ; area of traingle.

$$=\tfrac{1}{2}bc \sin \mathrm{A}=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\mathrm{R}= \frac{a}{2 \sin \mathrm{A}} \;;\; r= \frac{\Delta}{s} =(s-a) \tan\frac{\mathrm{A}}{2} \;;$$

$$r_1= \frac{\Delta}{s-a} =s \tan \frac{\mathrm{A}}{2} \;;$$

$$\underset{\theta-\theta}{\mathrm{Lt}} \;\frac{\sin\theta}{\theta} =1.$$

First edition	...	*1930*
Second edition	...	*1932*
Third edition	...	*1937*
Fourth edition	...	*1939*
Fifth edition	..	*1940*
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THE PRINCIPAL FORMULÆ IN TRIGONOMETRY

I. Circumference of a circle of radius $r = 2\pi r$.

$\pi = 3{\cdot}14159\ldots\ldots \left[\text{Approximations are } \frac{22}{7} \text{ and } \frac{355}{113}\right]$.

$\frac{1}{\pi} = {\cdot}31831.$

A Radian $= 57^\circ\ 17'\ 44{\cdot}8''$ nearly.

Two right angles $= 180^\circ = 200^g = \pi$ radians.

Circular measure of an angle $= \frac{\text{arc}}{\text{radius}} \left(\text{or } \theta = \frac{l}{r}\right)$.

II. $\sin^2\theta + \cos^2\theta = 1;$

$\sec^2\theta = 1 + \tan^2\theta;$

$\text{cosec}^2\theta = 1 + \cos^2\theta$ $1+\cot^2\theta$ Ghamid

III.

$\theta =$	0	30°	45°	60°	90°	180°
$\sin\theta$	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$	0
$\cos\theta$	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$	-1

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}};\ \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$

$\sin 18^\circ = \frac{\sqrt{5}-1}{4};\ \cos 36^\circ = \frac{\sqrt{5}+1}{4}.$

IV. $\sin(-\theta) = -\sin\theta; \qquad \cos(-\theta) = \cos\theta.$

$\sin(90^\circ-\theta) = \cos\theta; \qquad \cos(90^\circ-\theta) = \sin\theta.$

$\sin(90^\circ+\theta) = \cos\theta; \qquad \cos(90^\circ+\theta) = -\sin\theta.$

$\sin(180^\circ-\theta) = \sin\theta; \qquad \cos(180^\circ-\theta) = -\cos\theta.$

$\sin(180^\circ+\theta) = -\sin\theta; \qquad \cos(180^\circ+\theta) = -\cos\theta.$

$\tan(180^\circ+\theta) = \tan\theta.$

Note—The above formulæ help to reduce the angle to the least possible magnitude.

$\sin(360+\theta) =$

V. (*a*) If $\sin\theta=0$ then $\theta=n\pi$.

If $\cos\theta=0$ then $\theta=(2n+1)\dfrac{\pi}{2}$.

(*b*) If $\sin\theta=\sin\alpha$, then $\theta=n\pi+(-1)^n\alpha$.
If $\cos\theta=\cos\alpha$, then $\theta=2n\pi\pm\alpha$.
If $\tan\theta=\tan\alpha$, then $\theta=n\pi+\alpha$.
In all these n is 0, +ve, or —ve integer.

Note.—The above formulae show that the argument of a given trigonometrical ratio whose value is known is many valued.

VI.

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B.$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$

$$\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

$$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$

$$\cos P-\cos Q=2\sin\frac{P+Q}{2}\sin\frac{Q-P}{2}.$$

$$2\sin A\cos B=\sin(A+B)+\sin(A-B).$$
$$2\cos A\sin B=\sin(A+B)-\sin(A-B).$$
$$2\cos A\cos B=\cos(A+B)+\cos(A-B).$$
$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$
$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$$
$$\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$$
$$=\cos^2 B-\sin^2 A.$$

$$\sin 2A = 2\sin A \cos A \qquad (1)$$

$$= \frac{2\tan A}{1+\tan^2 A} \qquad (2)$$

$$\cos 2A = \cos^2 A - \sin^2 A \ldots\ldots (1)$$

$$= 2\cos^2 A - 1 \ldots\ldots (2)$$

$$= 1 - 2\sin^2 A \ldots\ldots (3)$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A} \ldots\ldots (4)$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\sin 3A = 3\sin A - 4\sin^3 A.$$

$$\cos 3A = 4\cos^3 A - 3\cos A.$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}\,;\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\sin A + \cos A = \pm\sqrt{(1+\sin 2A)}.$$

$$\sin A - \cos A = \pm\sqrt{(1-\sin 2A)}.$$

I. $\log_a 1 = 0$, $\log_a a = 1$.

$\log_a (mn) = \log_a m + \log_a n.$

$\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n.$

$\log_a m^n = n \log_a m$

$\log_a m = \log_b m \times \log_a b$. (Base changing formula).

$\log_b a = \dfrac{1}{\log_a b}.$

IX. If A, B, C are angles of a triangle and a, b, c the responding opposite sides, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \quad \text{(Sine Formulæ).}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \text{ etc.} \quad \text{(Cosine Formulæ).}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$a=b\cos C+c\cos B$ etc. (Projection Formulæ).

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}. \quad \text{(Napier's Anology)}$$

$\triangle$ = area of triangle $=\sqrt{s(s-a)(s-b)(s-c)}=\frac{1}{2}bc\sin A$

$=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C.$

X. Circum-radius of $\triangle$ ABC is

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\triangle}.$$

In-radius $r=\dfrac{\triangle}{s}=(s-a)\tan\dfrac{A}{2}$, etc.

e-radius opposite angle A is $r_1=\dfrac{\triangle}{s-a}=s\tan\dfrac{A}{2}$

$$=(s-c)\cot\frac{B}{2}=(s-b)\cot\frac{C}{2}$$

$$r_2=\frac{\triangle}{s-b}=s\tan\frac{B}{2}=(s-a)\cot\frac{C}{2}=(s-c)\cot\frac{A}{2}$$

$$r_3=\frac{\triangle}{s-c}=s\tan\frac{C}{2}=(s-a)\cot\frac{B}{2}=(s-b)\cot\frac{A}{2}.$$

Area of a quadrilateral inscribable in a circle

$$=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

XI. $\displaystyle\operatorname*{Lt}_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, when θ is measured in radians.

Area of a circle of radius $r=\pi r^2$.
Area of a sector of a circl

$$=\frac{\text{Arc}\times\text{Radius}}{2}.$$

CHAPTER I

MEASUREMENT OF ANGLES

1. The word 'Trigonometry' is derived from two Greek words—'trigonon' (meaning a triangle) and 'metron' (meaning 'I measure') and hence it literally means 'the measurement of triangles.' Originally, therefore, Trigonometry was that branch of Mathematics which had for its aim the measuring of the sides and the angles of a triangle and the investigation of the various relations which exist among them. But in modern times it has a wider application. It is no longer restricted to the solution of triangles; rather it comprises all investigations regarding angles in general, whether those angles are parts of a triangle or not.

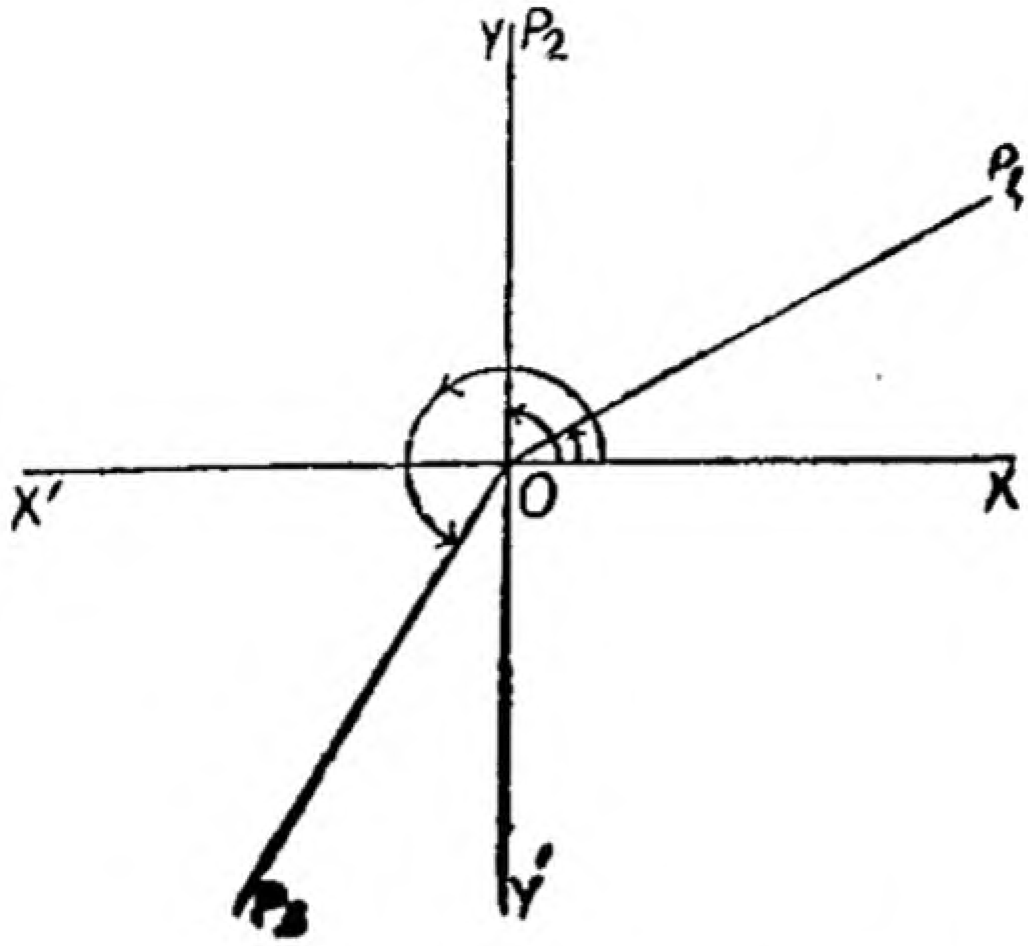
2. Angle of any Magnitude. The reader is already acquainted with a certain notion of an angle. According to Euclid an angle is the inclination between two intersecting straight lines. This is a very narrow definition and does not admit of any angle being greater than two right angles. The modern conception of an angle is different. *It is the amount of revolution, when a line revolving about one of its extremities, passes from one position to another.* Thus let X'OX and Y'OY be two straight lines at right angles to each other, and let a revolving line OP, originally coincident with the initial line OX, begin to revolve in the counter-clockwise direction and occupy the different positions as shown in the diagram.

When OP is at

(*i*) OP_1, the angle described is XOP_1
(*ii*) OP_2, ,, ,, XOP_2 (= rt. $\angle$)
(*iii*) OP_3, ,, ,, XOP_3 and so on.

When the line OP, in course of its revolution, coincides once more with OX, it has described an angle equal to four right angles. But suppose after making one complete revolution, it still continues to revolve. Now it will be describing angles greater than four right angles. Also it might continue revolving for any length of time. Thus an

angle may be of any magnitude, depending upon the number of revolutions made.



But there are two directions in which a line may revolve (*i*) clockwise and counter-clockwise. When revolving in the latter direction, it is said to describe positive angles; and when in the former direction, it is said to describe negative angles.

Thus angles may be *positive or negative* depending upon the *direction* of revolution, and may be great or small, depending upon the *number* of revolutions made.

Definitions. The lines X'OX and Y'OY divide the plane into four parts XOY, YOX' X'OY' and Y'OX called the *first*, the *second*, the *third* and the *fourth quadrants* respectively.

In the first quadrant the angle varies from	0° to 90°
,, second ,, ,, ,,	90° to 180°
,, third ,, ,, ,,	180° to 270°
and,, fourth ,, ,, ,,	270° to 360°

Thus it is clear that after revolving through an angle of 360° the revolving line arrives at the position from which it started. It is clear, therefore, that after describing angles of 30°, 390°, or 750° the revolving line has the same position.

Ex. 1. Find the position of revolving line after describing an angle of (*i*) 790°, (*ii*) −140°, (*iii*) −410°.

(*i*) Since $790 = 2 \times 360 + 70$, the revolving line has to make two complete revolutions and then turn through 70°. Thus the revolving line is in the first quadrant.

(*ii*) Here the revolving line has to revolve through 140° in the negative direction. Thus the revolving line is in the third quadrant.

(*iii*) Since $-410° = -(360 + 50)$, the revolving line has to make one complete revolution in the negative direction and then turn through 50° in the negative direction. Thus the revolving line is in the fourth quadrant.

The student is advised to draw the three figures separately.

Ex. 2. In which quadrants do the following angles lie ?—

(*i*) 300°. (*ii*) −220°. (*iii*) 770°. Ans. (*i*) Fourth (*ii*) Second (*iii*) First.

Ex. 3. In which quadrants do the following angles lie ? Also show these angles in a figure. (*i*) 450°. (*ii*) −810°. (*iii*) 270°. (*iv*) 970°.

3. Measurement of angles.

To measure any quantity, we are to find out how often it contains the unit of measurement.

In Trigonometry there are three systems of measuring angles :

(*a*) The *Sexagesimal* or the English System.
(*b*) The *Centesimal* or the French System.
(*c*) The system of *Circular* Measure.

(*a*) In the English or the *Sexagesimal* system, a right angle is divided and sub-divided into smaller parts as shown below :

1 rt. angle $= 90$ degrees (written as $90°$).
1 degree or $1° = 60$ minutes (written as $60'$).
1 minute or $1' = 60$ seconds (written as $60''$).

(*b*) In the French or the *Centesimal* system, a right angle is divided and sub-divided as shown below :

1 rt. angle $= 100$ grades (written as 100^g).
1 grade or $1^g = 100$ minutes (written as $100^\backprime$).
1 minute or $1^\backprime = 100$ seconds (written as $100^{\backprime\backprime}$).

The minutes and seconds used in this system are different from those used in the *Sexagesimal* system.

Observation.

From the above it is clear that the connecting link between these systems is a right angle and it is possible to convert the measure of an angle from one system to the other.

Ex. 1. Express $43^\circ\ 32'\ 15''$ in the French system.

$15'' = \frac{1}{4}$ minute $= 0{\cdot}24'$.

$$32'\ 15'' = 32{\cdot}25' = \frac{32{\cdot}25}{60} \text{ degrees} = 0{\cdot}5375^\circ$$

$$43^\circ\ 32'\ 15'' = 43{\cdot}5375^\circ = \frac{43{\cdot}5375}{90} \text{ rt. angle.}$$

$= {\cdot}48375$ of a right angle.

$= {\cdot}48375 \times 100 = 48{\cdot}375$ grades

$= 48^g\ 37{\cdot}5^{\backprime}$

$= 48^g\ 37^{\backprime}\ 50^{\backprime\backprime}$.

Ex. 2. Express $72^g\ 56^{\backprime}\ 25^{\backprime\backprime}$ in the English system.

$25^{\backprime\backprime} = {\cdot}25^{\backprime}$

$56^{\backprime}\ 25^{\backprime\backprime} = 56{\cdot}25' = {\cdot}5625^g$.

$72^g\ 56^{\backprime}\ 25^{\backprime\backprime} = 72{\cdot}5625^g = {\cdot}725625$ rt. angle.

$= {\cdot}725625 \times 90$ degrees

$= 65{\cdot}30625$ degrees

$= 65^\circ + {\cdot}30625 \times 60'$

$= 65^\circ + 18{\cdot}375'$

$= 65^\circ\ 18' + {\cdot}375 \times 60''$

$= 65^\circ\ 18'\ 22{\cdot}5''$.

Ex. 3. *Show that the ratio of Sexagesimal minutes in any angle to the Centesimal minutes in the same angle is* 27 : 50.

(P. U.)

Here let the Sexagesimal minutes in any angle $= x$, and Centesimal minutes in the same angle $= y$.

Number of rt. angles in the first case $= \dfrac{x}{60 \times 90}$ and in the second case $= \dfrac{y}{100 \times 100}$

$$\therefore \quad \frac{x}{60 \times 90} = \frac{y}{100 \times 100}$$

$$i.e.,\ \frac{x}{y}=\frac{54}{100}$$

$i.e.,\ x : y :: 27 : 50.$

(*c*) **Circular System.** This system is used in all the higher branches of Mathematics. The unit taken in it is called a **Radian** which is defined as *th angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.*

But before we take a radian to be a unit, we must show that it is a constant quantity, *i.e.*, it retains the same value whatever the radius of the circle may be.

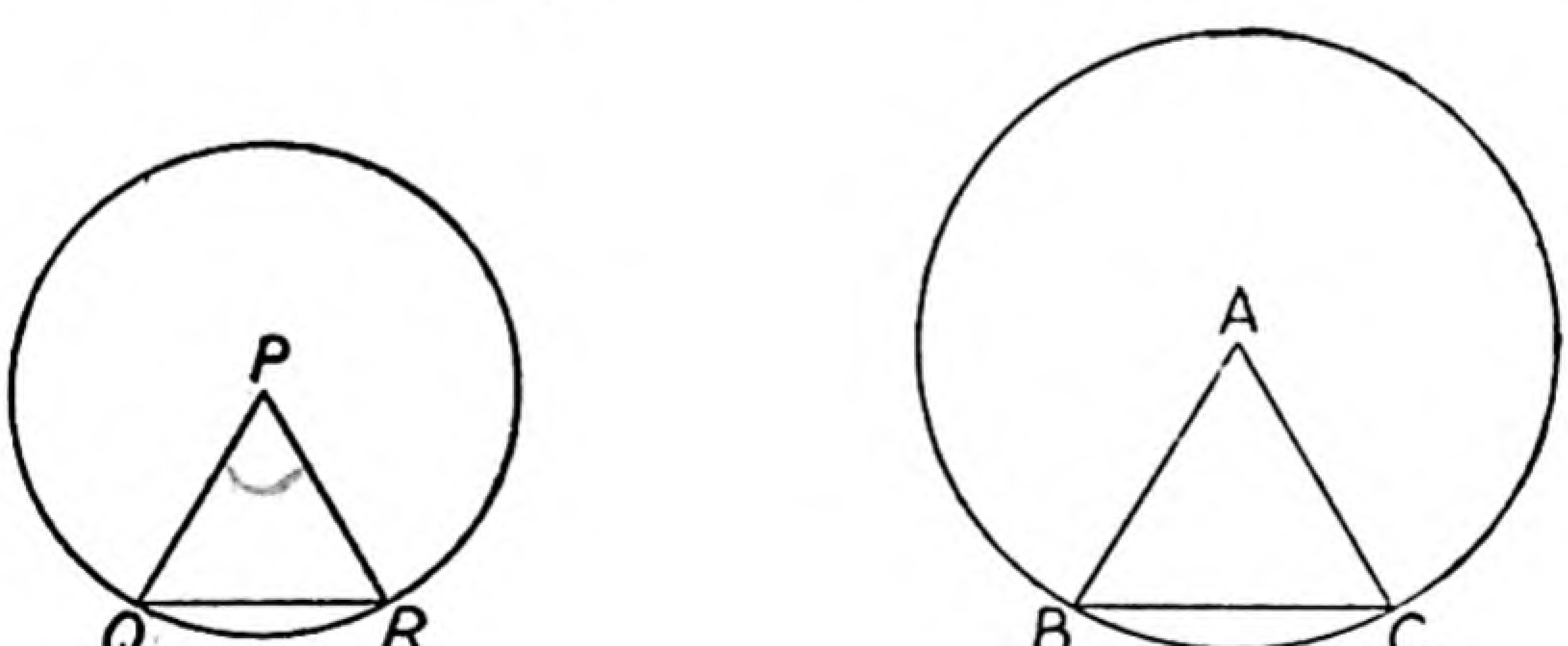
This we shall now show in the next two articles.

4. The circumference of a circle bears a constant ratio to its diameter.

Take any two circles whose radii are r and **R** and whose centres are P and A; and in each circle let a regular polygon of n sides be described.

Let QR be a side of the first and BC a side of the second and let their lengths be p and a respectively. Join QP, RP, BA and CA.

Now $\angle QPR=\frac{1}{n}$ of four right angles $=\angle BAC$ and



$\frac{QP}{BA}=\frac{r}{R}=\frac{PR}{AC}$ so that Δs PQR and ABC are similar :

$$\therefore\quad \frac{QR}{QP}=\frac{BC}{BA} \qquad \therefore\quad \frac{nQR}{QP}=\frac{nBC}{BA},\ i.e.,\ \frac{np}{r}=\frac{na}{R}$$

or $\frac{p_1}{2r}=\frac{p_2}{2R}$ where p_1 and p_2 are the perimeters of

the polygons. This is true whatever the number of sides of the polygons. Now by taking n sufficiently large we can make the perimeter of the two polygons differ from the circumferences of the corresponding circles by as small a quantity as we please, so that ultimately,

$\frac{c}{2r}=\frac{C}{2R}$ where c and C are the circumferences of the two circles.

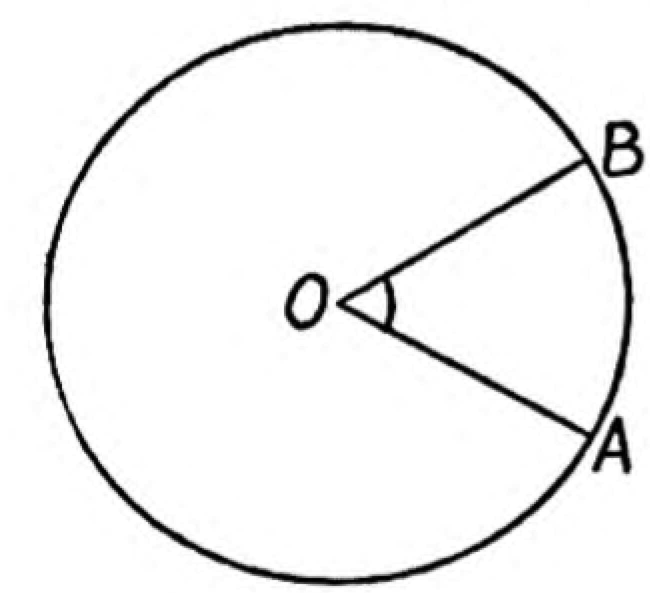
Hence the ratio of the circumference of a circle to its diameter is constant.

The constant ratio is usually denoted by π, so that if r be the radius of a circle, its circumference is $2\pi r$. The value of π cannot be stated exactly because it is an unending non-recurring decimal.

However some of its approximate values are $\frac{22}{7}$, $\frac{355}{113}$, 3·14159...

Also $\frac{1}{\pi}=\cdot 3183$, correct to four places of decimals.

5. **The radian is a constant angle.**



Let the *arc* AB of the circle be equal to the radius OA. Then by definition $\angle AOB=1$ radian.

Since the arcs of a circle are to each other as the angles which they subtend at the centre, we have

$$\frac{\text{arc AB}}{\text{circumference}}=\frac{\angle AOB}{4 \text{ rt. angles}}$$

i.e., $$\frac{OA}{2\pi.OA}=\frac{1 \text{ radian}}{4 \text{ rt. angles}}.$$

1 radian $=\frac{2 \text{ rt. angles}}{\pi}$, which is a constant quantity.

Thus **1 Radian** $=\frac{180}{\pi}$ **Degrees and** $1^\circ=\frac{\pi}{180}$ **Radians.**

It follows that a radian $=\frac{180^\circ}{\pi}=\frac{180}{3\cdot 1416}$

$=57\cdot 296^\circ=57^\circ 17' 45''$,

or $206265''$ correct to the nearest second.

Cor. π **radians = 2 right angles = $180° = 200^g$.**

6. Definition. The Circular Measure *of an angle is the number of radians it contains.*

For example $30° = 30° \times \frac{\pi}{180} = \frac{\pi}{6}$ radian. Thus the circular measure of an angle of $30°$ is $\frac{\pi}{6}$.

Similarly $60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ radian and the circular measure of an angle of $60°$ is $\frac{\pi}{3}$.

Ex. 1. Find the circular measure of the following :— (*i*) $90°$, (*ii*) $180°$, (*iii*) $-360°$.

Ans. (*i*) $\frac{\pi}{2}$, (*ii*) π, (*iii*) -2π.

Ex. 2. Find the number of degrees in the angles whose circular measures are (*i*) 2π, (*ii*) $\frac{5\pi}{4}$ (*iii*) $\frac{3\pi}{5}$ (*iv*) $\frac{\pi}{6}$.

Ans. (*i*) 360, (*ii*) 225, (*iii*) 108. (*iv*) 30.

Ex. 3. Find the circular measure of (*i*) $5° \; 37' \; 30''$. (*ii*) $1^g \, 1'$.

(*i*) $5° \; 37' \; 30'' = \frac{1}{16}$ of a right angle.

Now $\because$ 2 rt. angles $= \pi$ radians,

$$\therefore \text{ 1 right angle} = \frac{\pi}{2} \text{ radians}$$

$$\text{and } \tfrac{1}{16} \text{ rt. angle} = \frac{\pi}{32} \text{ of a radian.}$$

$$(ii)\; 1^g \, 1' = \cdot 0101 \text{ rt. angle} = \frac{\pi}{2} \times \cdot 0101 \text{ radian}$$

$$= \cdot 00505\, \pi \text{ of a radian.}$$

Ex. 4. Express $2\cdot 2$ radians in the (*i*) Sexagesimal (*ii*) Centesimal systems.

(*i*) π radians $= 180°$ $\qquad \therefore$ one radian $= \frac{180°}{\pi}$

$\therefore$ 2·2 radians $= \dfrac{180}{\pi} \times 2{\cdot}2$ degrees.

$= 180 \times \dfrac{7}{22} \times \dfrac{22}{10}$ degrees $= 126°$.

(*ii*) π radians $= 200$ grades; $\therefore$ one radian $= \dfrac{200^g}{\pi}$

$\therefore$ 2·2 radians $= \dfrac{200}{22} \times 7 \times \dfrac{22}{10}$ grades $= 140^g$.

Ex. 5. Express in circular measure, and also in degrees the angle of a regular polygon of 40 sides ($\pi = \frac{22}{7}$).

The sum of all the external angles being 360°, each external angle, therefore, is $\dfrac{360°}{40} = 9°$.

Hence an angle of the polygon is $180° - 9° = 171°$.

Now $180° = \pi$ radians

$\therefore \quad 1° = \dfrac{\pi}{180}$ radian

$\therefore \quad 171° = \frac{171}{180} \times \pi$ radians

$= \frac{171}{180} \times \frac{22}{7}$ radians $= 2\frac{69}{70}$ radians.

Ex. 6. Express in circular measure as well as in degrees the angle of a regular polygon of 15 sides.

The sum of the fifteen exterior angles of the polygon is 360°; therefore each of these angles is 24° so that each interior angle is of 156°.

Now $180° = \pi$ radians.

$\therefore \quad 156° = \dfrac{\pi}{180} \times 156 = \dfrac{13}{15}\,\pi$ radians.

Therefore the interior angle of a regular polygon of 15 sides is 156° or $\frac{13}{15}\,\pi$ of a radian.

Ex. 7. The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as 45 is to π. Find the angles in degrees.

Let the angles be $(a-d)°$, $a°$, $(a+d)°$.

The sum of the three angles being 180°,

$\therefore \ (a-d) + a + (a+d) = 180°$,

or $a = 60$ so that the angles are $(60-d)°$, $60°$, $(60+d)°$.

Now $180° = \pi$ radians

$\therefore \quad 1° = \dfrac{\pi}{180}$ radian

and $\therefore \quad (60+d)^\circ = \frac{\pi}{180}(60+d)$ radians.

Now therefore from the given data,

$$\frac{45}{\pi} = \frac{60-d}{\frac{\pi}{180}(60+d)}, \text{ or } 180\ (60-d)=45(60+d)$$

$\therefore\ 4(60-d)=60+d$

$\therefore 5d=180$, so that $d=36$

Hence the angles are $24^\circ, 60^\circ, 96^\circ$.

Note.—θ radians is written as θ^c where c is the first letter of the word Circular Measure. But when the unit used is a radian, it is customary not to mention it. Thus an angle π means an angel of π radians.

7. *To covert from Sexagesimal or Centesimal measure to Circular measure or vice-versa.*

Let D be the number of degrees, G the number of grades and C the number of radians contained in an angle.

$\because$ the ratio of an angle to a right angle is the same in each system of measurement,

$$\frac{D}{90}=\frac{G}{100}=\frac{C}{\pi/2} \quad \therefore \quad \frac{D}{9}=\frac{G}{10}=\frac{20C}{\pi}$$

$$\therefore \quad D=\frac{180C}{\pi} \text{ and } C=\frac{\pi}{180}D.$$

In practice it is better to express the angle as a fraction of a right angle and then proceed to the desired system from the right angle.

EXERCISE 1

1. Find the quadrants in which the following angles lie :—

$(i)\ 815^\circ.\ (ii)\ -275^g.\ (iii)\ -\frac{3\pi}{4}.\ (iv)\ \frac{5\pi}{16}.$

2. Express the following angles in grades, minutes and seconds in (Centesimal System).

$(i)\ 69^\circ\ 13'\ 30''\ \ (ii)\ \frac{5\pi^c}{12}.$

3. Find the circular measure of

$(i)\ 15^\circ\ (ii)\ 15^g\ (iii)\ 135^\circ\ (iv)\ 135^g.$ (P. U.

4. The angles of a triangle are 4 : 5 : 6. Find thei circular measure.

5. The angles of a triangle are to one another in the ratio 2 : 3 : 4 ; express them in circular measure as well as in degrees.

6. Express in radians the vertical angle of an isosceles triangle which has each angle at the base double the third angle.

7. The angles of a triangle are $4x$ degrees, $5x$ degrees and $10x$ grades ; find them all in degrees, and also in grades.

8. The angles of a triangle are in A. P. and the greatest is double the least ; express the angles in degrees, radians and grades.

9. The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as $60 : \pi$. Find the angles.

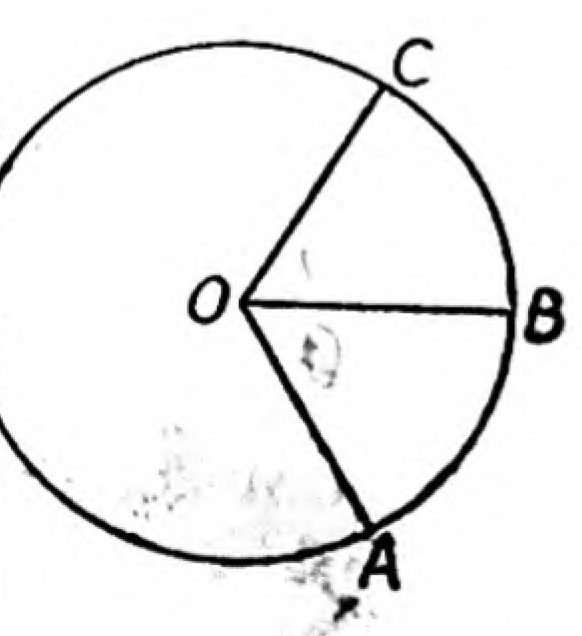
10. The angles of a quadrilateral are in A. P. and the greatest is double the least. Express the least angle in radians.

11. Express in circular measure and also in degrees the angle of a regular polygon of n sides.

12. An angle is such that the difference of reciprocals of the measure of grades and degrees in it multiplied by 2π is equal to its circular measure ; find the angle in degrees.

8. The length of an arc of a circle of a given radius can be expressed in terms of the circular measure of the angle subtended by it at the centre and conversly.

Let AB be an arc of length l, of a circle (centre O and radius r) and let θ be the circular measure of the angle AOB.



Let the arc BC$=r$ so that $\angle$BOC $=$one radian.

As the arcs of a circle are in the ratio of angles subtended at the centre

$$\frac{\angle AOB}{\angle BOC}=\frac{\text{arc } AB}{\text{arc } BC}$$

or $$\frac{\theta \text{ radians}}{1 \text{ radian}}=\frac{l}{r}, \quad \therefore \theta=\frac{l}{r}.$$

It follows that the circular measure of an angle

$$= \frac{\textbf{length of the arc}}{\textbf{radius of the circle}}.$$

Also $l=r\theta=$radius of the circle$\times$circular measure of angle subtended.

Note. Notice that if the circle be of unit radius, then the circular measure of an angle at the centre is equal to the length of the arc subtending it.

Ex. 1. Express in radians and degrees the angle subtended at the centre of a circle by an arc whose length is 15 inches, the radius of the circle being 25 inches.

Number of radians in the angle

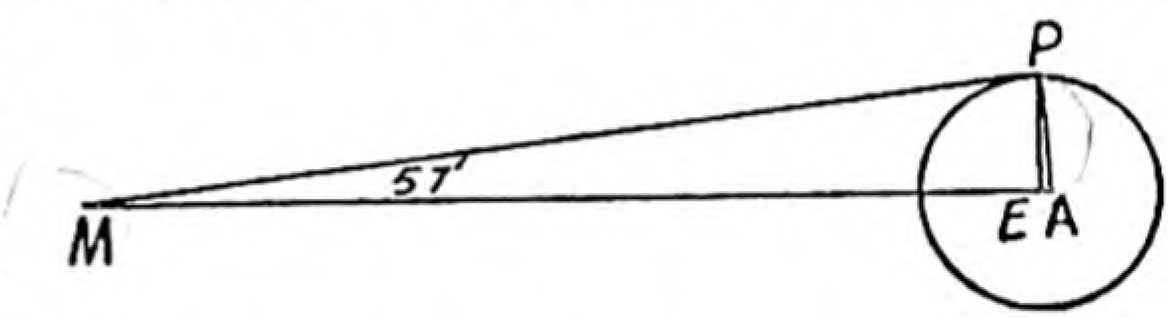
$$=\frac{\text{length of arc}}{\text{radius}}=\frac{15}{25}=\frac{3}{5}.$$

Also π radians $=180°$

$$\therefore\quad \frac{3}{5}\text{ radian}=\frac{180°}{\pi}\times\frac{3}{5}=\frac{180\times7}{22}\times\frac{3}{5}\text{ degrees}$$
$$=34°\ 22'\ 38\,9''.$$

Ex. 2. Assuming that the earth's radius is 3960 miles and that it subtends an angle 57′ at the centre of the moon, find the distance of the moon from the earth's centre.

Let M be the centre of the moon and PE the radius of the earth (centre E.) With M as centre and MP as radius draw an arc cutting ME produced in A. When the angle at M is small, A and E are very near each other and the lengths PE and PA may be taken as approximately equal and so also the lengths ME and MA.



Now we know that in any circle $\theta=\frac{l}{r}$ where θ is the circular measure of the angle subtended at the centre by an arc of length l, and r is the radius of the circle. Hence

$$r=\frac{l}{\theta}.$$

But $l=3960$ miles, $\theta=57'=\frac{57}{60}\times\frac{\pi}{180}$ radians.

$$\therefore\quad r=\frac{3960\times7\times60\times180}{57\times22}=238737\text{ miles nearly.}$$

EXERCISE II

1. In a circle of 5 feet radius what is the length of the arc which subtends an angle of 30° 15′ at the centre ?

2. The diameter of a graduated circle is 6 feet and the

graduations on its rim are $5'$ apart; find the actual distance from one graduation to another.

3. The perimeter of a certain sector of the circle is equal to half that of the circle of which it is a sector. Find the circular measure of the angle of the sector. (P. U.)

4. What is the ratio of the radii of two circles at the centre of which two arcs of the same lengths subtend angles of 60° and 75°?

5. At what distance does a man, whose height is 6 feet subtend an angle of $10'$? (M. U.)

6. Find the length which at a distance of one mile will subtend an angle of $1'$ at the eye. (B. U.)

7. A circular wire of radius 3 inches is cut and then bent so as to lie along the circumference of a hoop whose radius is 4 ft. Find in radians the angle which it subtends at the centre of the hoop. (P. U. 1938)

8. If the diameter of the moon subtends an angle of $30'$ at the eye of the observer, and the diameter of the sun an angle of $32'$, and if the distance of the sun be 375 times the distance of moon, find the ratio of their diameters.

9. The diameter of the moon is $30'$; find how far from the eye a coin of $\frac{1}{2}$ inch radius must be held so as to hide the moon.

FORMULÆ OF CHAPTER I

(*a*) *Sexagesimal System*	(*b*) *Centesimal System*
One rt. angle $=90^\circ$	**1 rt. angle** $=100^g$
$1^\circ=60'$	$1^g=100`$
$1'=60''$	$1`=100``$.

(*c*) *Circular Measure of an angle = No of Radians contained by it.*

$$1 \text{ Radian} = \frac{180}{\pi} \text{ Degrees}$$

$$1 \text{ Degree} = \frac{\pi}{180} \text{ Radian}$$

(*d*) $\frac{D}{90} = \frac{G}{100} = \frac{\pi C}{2}$

(*e*) 1 rt. angle $=90^\circ=100^g$.

(*f*) **Circular Measure of an angle** $=$

$$\frac{\textbf{Length of Arc}}{\textbf{Radius of the Circle}}.$$

REVISION QUESTIONS I

1. Find the angle in radians through which the minute hand of a watch turns in an interval of 25 minutes.

2. Define the circular measure of an angle ; find the circular measure of 1° and $1'$ to five places of decimals. (B. U.)

3. In a right angled triangle the difference between the acute angles is $\frac{\pi}{9}$ in circular measure. Express the angles in degrees.

4. The angles of a triangle are in A. P. ; one of them being 95°, find all the three angles in radians.

5. The circular measures of two angles of a triangle are $\frac{1}{2}$ and $\frac{2}{7}$ respectively. Find the number of degrees in the third angle ($\pi=\frac{22}{7}$).

6. One angle of a triangle is $\frac{3\pi}{10}$ radians, the other is 70 grades ; find the third in degrees.

7. A, B, C are the angles whose magnitudes are 30°, 60^g and $\frac{8\pi}{15}$ radians respectively. Show that a triangle can be formed having these angles.

8. Express in radians the fourth angle of a quadrilateral which has the three angles $46^\circ\,30'\,10''$; $75^\circ\,44'\,45''$; $123^\circ\,9'\,35''$ respectively.

9. Find the number of sides of a regular polygon each of whose actue angles is $\frac{3\pi}{4}$ radians.

10. Calculate the length of an arc of a circle of radius 2 feet, which subtends an angle of 1·1 radians at the centre of the circle. Take $\pi=\frac{22}{7}$.

11. Find the length of an arc which subtends an angle

of 5 $''$ at the centre of a circle whose radius is 4000 miles.

12. If the circumference of a circle be divided into five parts which are in Arithmetical Progression and if the greatest part be six times the least, find in radians the angles that the parts subtend at the centre of the circle.

13. The diameter of the sun subtends an angle of 32$'$ at the eye of an observer ; show that the diameter of the sun is 866,000 miles approximately, assuming that the distance of the sun from the earth is 93,000,000 miles.

14. The radius of a certain circle is 3 feet ; find approximately the length of an arc of this circle, if the length of the chord of the arc be 3 feet also.

15. A train is travelling at the rate of 10 miles per hour on a curve of half a mile radius. Through what angle has it turned in one minute ?

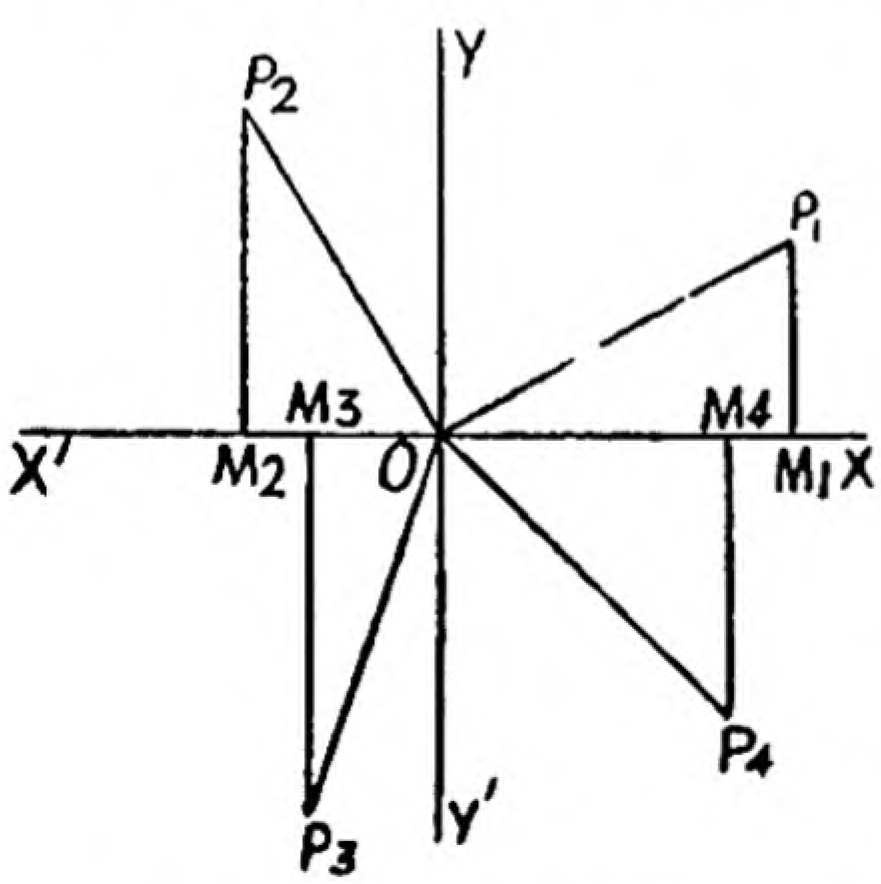
16. If M, S, m, s, denote respectively the number of English minutes and seconds, and French minutes and seconds in any angle, prove that

$$\frac{M}{27}=\frac{m}{50} \text{ and } \frac{S}{81}=\frac{s}{250}.$$

CHAPTER II

TRIGONOMETRICAL RATIOS

9. Sign Convention for Lines.



Let X$'$OX and Y$'$OY be two fixed lines at right angles. We shall adopt the following convention :

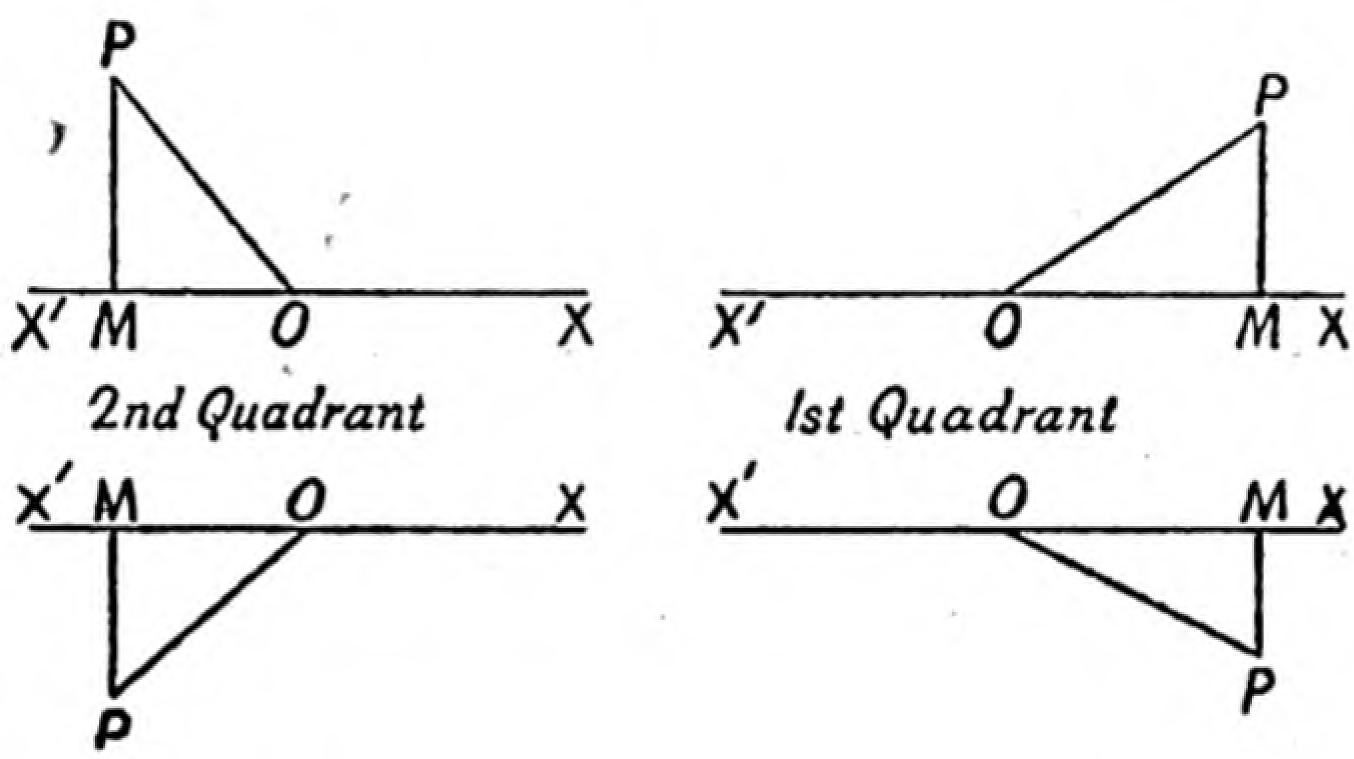
Distances measured from O in the direction OX shall be regarded as positive and shall be denoted by positive numbers, and distances measured in the direction OX$'$ shall be regarded as negative and shall be denoted by negative numbers. Similarly distances measured in the direction OY shall be regarded as positive and those in the direction OY$'$ as negative, *i.e.*, distances measured upwards at right angles to X$'$OX shall be positive and those measured downwards shall be negative. Thus in the above figure, OM_1, OM_4 are positive while OM_2

and OM_3 are negative ; and M_1P_1, M_2P_2 are positive while M_3P_3 and M_4P_4 are negative.

The revolving line is always regarded as positive ; in the above figures, for example, OP_1, OP_2, OP_3 OP_4 are all regarded as positive.

10. Definitions of Trigonometrical Ratios.

Let OX be the initial line ; let OP, the revolving line, originally coincident with the initial line, begin to revolve about O in either direction and describe an angle XOP ($=\theta$, say). From any point P on the final position of the revolving line, draw PM Perpendicular OX, meeting OX, produced if necessary, in M.



2nd Quadrant Ist Quadrant

3rd Quadrant 4th Quadrant

(1) $\dfrac{MP}{OP}$ is called the sine of the angle θ ;

(2) $\dfrac{OM}{OP}$,, ,, cosine ,, ,, θ ;

(3) $\dfrac{MP}{OM}$,, ,, tangent ,, ,, θ ;

(4) $\dfrac{OM}{MP}$,, ,, cotangent ,, ,, θ ;

(5) $\dfrac{OP}{OM}$,, ,, secant ,, ,, θ ;

and (6) $\dfrac{OP}{MP}$,, ,, cosecant ,, ,, θ ;

To these six ratios it is usual to add two more (7), $1-\frac{OM}{OP}$ is called the versed sine and (8), $1-\frac{MP}{OP}$ is called the coversed sine of the angle θ; but these two are seldom used.

These are abbreviated into

$$\sin\theta = \frac{MP}{OP};\ \cos\theta=\frac{OM}{OP};\ \tan\theta=\frac{MP}{OM};$$

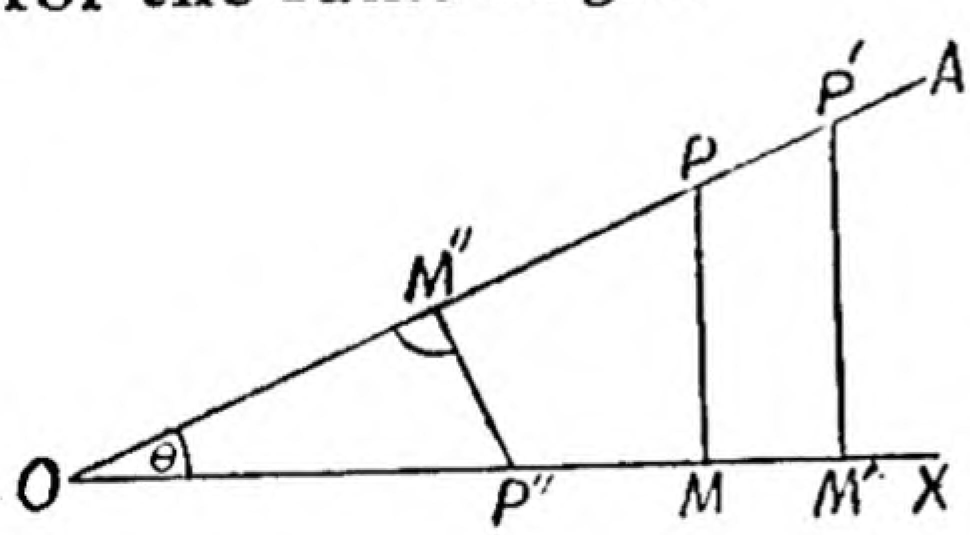
$$\cot\theta = \frac{OM}{MP},\ \sec\theta=\frac{OP}{OM};\ \text{cosec}\ \theta=\frac{OP}{MP},$$

vers $\theta=1-\frac{OM}{OP}$; and covers$\theta=1-\frac{MP}{OP}$.

Note 1.—The student must notice that $\sin\theta$ does not mean $\sin\times\theta$. Sin or sine without an angle has got no meaning. $\sin\theta$ is to be taken as a whole, denoting as it does, a certain ratio. The same remark applies to the other trigonometrical ratios.

Note 2.—The trigonometrical ratios defined above are also called circular functions.

11. The Trigonometrical Ratios are always the same for the same angle.



Take another point P′ on the revolving line OPA and draw PM and P′M′ perpendiculars to the initial line OX. Triangles OPM and OP′M′ are equiangular;

$\therefore$ their corresponding sides are proportional.

Hence $\frac{M'P'}{OP'}=\frac{MP}{OP}=\sin\theta$.

Similarly $\frac{OM'}{OP'}=\frac{OM}{OP}=\cos\theta$ and $\frac{M'P'}{OM'}=\frac{MP}{OM}=\tan\theta$.

Again if OX be taken as the revolving line and OA the initial line and P″ be a pt. in OX so that P″M″ is perpendicular to OA, then in the rt. angled Triangles OPM and OP″M″,

$\angle MOP=\angle M''OP''$

$\therefore$ the Triangles are equiangular.

Hence $\frac{M''P''}{OP''}=\frac{MP}{OP}=\sin\theta.\ \frac{OM''}{OP''}=\frac{OM}{OP}=\cos\theta$

and $\dfrac{M''P''}{OP''}=\dfrac{MP}{OP}=\tan$.

Thus each of the trigonometrical ratios depends only upon the magnitude of the angle θ and not upon the absolute length of OP. These are also independent of the fact whether P is taken on one arm or on the other of the angle.

Note.—It is customary to denote the positive integral powers of trigonometrical ratios thus :

$(\sin\theta)^2$ is denoted by $\sin^2\theta$ and is read as 'sine square θ.'

$(\sin\theta)^3$ is denoted by $\sin^3\theta$ and is read as 'sine cubed θ.' And so on for the other ratios.

But $(\sin\theta)^{-1}$ is never written as $\sin^{-1}\theta$. This latter notation has got different meaning which will be explained in Chapter IX.

12. Some Important Relations.

$$\left.\begin{aligned}&\sin\theta\ \text{cosec}\ \theta=\frac{MP}{OP}\cdot\frac{OP}{MP}=1\\ \therefore\quad &\sin\theta=\frac{1}{\text{cosec}\ \theta}\ \text{and cosec}\ \theta=\frac{1}{\sin\theta}.\end{aligned}\right\}A$$

$$\left.\begin{aligned}&\cos\theta.\ \sec\theta=\frac{OM}{OP}\cdot\frac{OP}{OM}=1,\\ \therefore\quad &\cos\theta=\frac{1}{\sec\theta}\ \text{and}\ \sec\theta=\frac{1}{\cos\theta}.\end{aligned}\right\}B$$

$$\left.\begin{aligned}&\tan\theta.\ \cot\theta=\frac{MP}{OP}\cdot\frac{OM}{MP}=1,\\ \therefore\quad &\tan\theta=\frac{1}{\cot\theta}\ \text{and}\ \cot\theta=\frac{1}{\tan\theta}.\end{aligned}\right\}C$$

12. (*a*) Fundamental Relations betwen the Trigonometrical Functions.

Let the revolving line, starting from the initial position OX trace out an angle θ. From P any point in the final position of the revolving line draw PM Perpendicular to OX.

Then from the rt. angled $\triangle$ OPM, we get

[See Figures page 19]

$MP^2+OM^2=OP^2$.

(*i*) dividing this equation by OP^2, we get

$$\left(\frac{MP}{OP}\right)^2+\left(\frac{OM}{OP}\right)^2=1, \text{ i.e., } \sin^2\theta+\cos^2\theta=1 \qquad \}D$$

(*ii*) dividing the same equation by OM^2 we get

$$\left(\frac{MP}{OM}\right)^2+1=\left(\frac{OP}{OM}\right)^2 \text{ i.e., } 1+\tan^2\theta=\sec^2\theta\,; \qquad \}E$$

(*iii*) dividing the very same equation by MP^2, we get

$$1+\left(\frac{OM}{MP}\right)^2=\left(\frac{OP}{MP}\right)^2 \text{ i.e., } 1+\cot^2\theta=\operatorname{cosec}^2\theta. \qquad \}F$$

Note.—The student should draw angle θ in all the four quadrants to prove the above relations to be universally true.

Ex. 1. Show that $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$.

Here $(\sin\theta+\cos\theta)^2=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$

$$=1+2\sin\theta\cos\theta.$$

Ex. 2. Prove that $\sqrt{\operatorname{cosec}^2 A-1}=\cos A\operatorname{cosec} A$

The left-hand side $=\sqrt{1+\cot^2 A-1}$

$$=\cot A=\frac{\cos A}{\sin A}=\cos A\operatorname{cosec} A.$$

Ex. 3. Prove that $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$.

The left-hand side $=\sqrt{\dfrac{(1-\sin A)(1-\sin A)}{(1+\sin A)(1-\sin A)}}$

$$=\frac{1-\sin A}{\sqrt{1-\sin^2 A}}=\frac{1-\sin A}{\cos A}=\frac{1}{\cos A}-\frac{\sin A}{\cos A}=\sec A-\tan A.$$

Ex. 4. Prove that

$$\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}=\sec A\operatorname{cosec} A+1.$$

The L. H. S. $=\dfrac{\dfrac{\sin A}{\cos A}}{1-\dfrac{\cos A}{\sin A}}+\dfrac{\dfrac{\cos A}{\sin A}}{1-\dfrac{\sin A}{\cos A}}$

$$=\frac{\sin^2 A}{\cos A(\sin A-\cos A)}+\frac{\cos^2 A}{\sin A(\cos A-\sin A)}$$

$$=\frac{\sin^3 A-\cos^3 A}{\sin A\cos A(\sin A-\cos A)}$$

$$=\frac{(\sin A-\cos A)(\sin^2 A+\cos^2 A+\sin A\cos A)}{\sin A\cos A(\sin A-\cos A)}$$

$$=\frac{1+\sin A\cos A}{\sin A\cos A}=\frac{1}{\sin A\cos A}+1$$

$$=\frac{1}{\sin A}\cdot\frac{1}{\cos A}+1=\sec A\operatorname{cosec} A+1.$$

Note.—It may be noticed that it is sometimes found convenient to express all the trigonometrical ratios in terms of the sine and cosine as in the above example.

Ex. 5. Prove that $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)$ is independent of θ.

Here $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)$
$=2(\sin^2\theta+\cos^2\theta)(\sin^4\theta+\cos^4\theta-\sin^2\theta\times\cos^2\theta)-3(\sin^4\theta+\cos^4\theta)$
$=-2\sin^2\theta\cos^2\theta-\sin^4\theta-\cos^4\theta=-(\sin^2\theta+\cos^2\theta)^2=-1.$

EXERCISE III

Prove the following identities :

1. $\sin^2 A-\cos^2 B=\sin^2 B-\cos^2 A.$

2. $\dfrac{1}{\operatorname{cosec}^2\theta}+\dfrac{1}{\sec^2\theta}=1.$

3. $(\sec^2\theta-1)\cot^2\theta=1.$ 4. $(1-\sin^2\theta)\sec^2\theta=1.$

4 5. $\tan\theta(1-\cot^2\theta)+\cot\theta(1-\tan^2\theta)=0$

Simplify the following expressions :—

6. $\dfrac{1}{\sin^2\theta}-1.$ 7. $\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$

8. $\dfrac{1}{1+\sin A}+\dfrac{1}{1-\sin A}.$ 9. $\dfrac{\sin^2 x}{\tan x}-\dfrac{\cos^2 x}{\cot x}.$

Prove the following identities :—

10. $\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cot\theta-1}{\cot\theta+1}.$

11. $\dfrac{1}{1+\cos\theta}+\dfrac{1}{1-\cos\theta}=2\left(1+\dfrac{\cos^2\theta}{\sin^2\theta}\right).$

12. $(\tan\theta+\sec\theta)^2=\dfrac{1+\sin\theta}{1-\sin\theta}.$

13. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$.

14. $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$.

15. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \operatorname{cosec}^2\theta$. (D. U. 1940)

16. $(\tan\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$
$= (1 + \sec\theta \operatorname{cosec}\theta)^2$.

17. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$.

18. $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.

19. $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$.

20. $\operatorname{cosec}^2\theta + \sec^2\theta = \operatorname{cosec}^2\theta \sec^2\theta$.

21. $\sec A - \tan A = \dfrac{\cos A}{1 + \sin A}$.

22. $\dfrac{1}{\cot A + \tan A} = \sin A \cos A$.

23. $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = 2 \operatorname{cosec}\theta$.

24. $\dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}$.

25. $\dfrac{\cot A + \tan B}{\tan A + \cot B} = \dfrac{\cot A}{\cot B}$.

26. $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$.

27. $\sqrt{\dfrac{1 + \cos\theta}{1 - \cos\theta}} = \operatorname{cosec}\theta + \cot\theta$.

28. $\sin^2\theta \cos^2\phi + \cos^2\theta \sin^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta \cos^2\phi = 1$.

29. $\sin^2 A(2 + \tan^2 A) = \sec^2 A - \cos^2 A$.

30. $\dfrac{\cos\theta}{1 - \tan\theta} + \dfrac{\sin\theta}{1 - \cot\theta} = \sin\theta + \cos\theta$.

31. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2\sec A \tan A + 2\tan^2 A$.

32. $(2\sin\theta\cos\theta)^2 + (\cos^2\theta - \sin^2\theta)^2 = 1$.

33. $(x\sin\theta + y\cos\theta)^2 + (x\cos\theta - y\sin\theta)^2 = x^2 + y^2$.

34. $\dfrac{\sec^2 A \sin^2 A - \operatorname{cosec}^2 A + \operatorname{cosec}^2 A \cos^2 A}{\sec^2 A \sin^2 A - \operatorname{cosec}^2 A \cos^2 A} = \sin^2 A$.

(B. U.)

35. $\frac{\cos A+\cos B}{\sin A-\sin B}+\frac{\sin A+\sin B}{\cos A-\cos B}=0.$

36. $(1-\tan\theta)^2+(1-\cot\theta)^2=(\sec\theta-\operatorname{cosec}\theta)^2.$

37. $\cot^4\theta+\cot^2\theta=\operatorname{cosec}^4\theta-\operatorname{cosec}^2\theta.$

38. $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta.$

39. If $\sin\theta+\cos\theta=a$, and $\tan\theta+\cot\theta=b$, prove that

$$\frac{a^2-1}{2}=\frac{1}{b}.$$

40. Can 0·6 and 0·8 be the sine and cosine respectively of one and the same angle? Can 0·7 and 0·9 be so?

12. (*b*) **Elimination.**—The fundamental relations established in Article 12 (*a*) are very helpful in eliminating θ from two given equations. The method will be clear from the following examples.

Ex. 1. Eliminate θ between

$$a\cos\theta+b\sin\theta+c=0$$
$$a'\cos\theta+b'\sin\theta+c'=0.$$

Solving for $\sin\theta$ and $\cos\theta$ we get

$$\frac{\cos\theta}{bc'-cb'}=\frac{\sin\theta}{ca'-ac'}=\frac{1}{ab'-a'b},$$

i.e., $\cos\theta=\frac{bc'-cb'}{ab'-a'b}$ and $\sin\theta=\frac{ca'-ac'}{ab'-a'b}.$

But $\sin^2\theta+\cos^2\theta=1$

$\therefore$ $(bc'-cb')^2+(ca'-ac')^2=(ab'-a'b)^2.$

Ex. 2. Eliminate θ between the equations

$$a\tan\theta+b\sec\theta=c, \qquad p\tan\theta-q\sec\theta=r.$$

Here solving for $\tan\theta$ and $\sec\theta$ we get

$$\tan\theta=\frac{cq+br}{aq+bp},\ \sec\theta=\frac{cp-ar}{bp+aq}.$$

Since $\sec^2\theta=1+\tan^2\theta$, we get

$$\left(\frac{cp-ar}{bp+aq}\right)^2=1+\left(\frac{cq+br}{aq+bq}\right)^2.$$

EXERCISE IV

Eliminate θ from the equations :—

1. $x=a\cos\theta,\ y=b\sin\theta.$ 2. $x=a\cos^2\theta,\ y=b\sin^2\theta.$

3. $p=\sin\theta-q,\ q=\cos\theta+p.$

4. $x=a\cos^n\theta,\ y=b\sin^n\theta.$

5. $\tan\theta+\sin\theta=x,\ \tan\theta-\sin\theta=y.$

6. $x^3=\tan^2\theta,\ y^3=\sec^2\theta.$

7. $x=3-\cot\theta,\ y=4+\operatorname{cosec}\theta.$

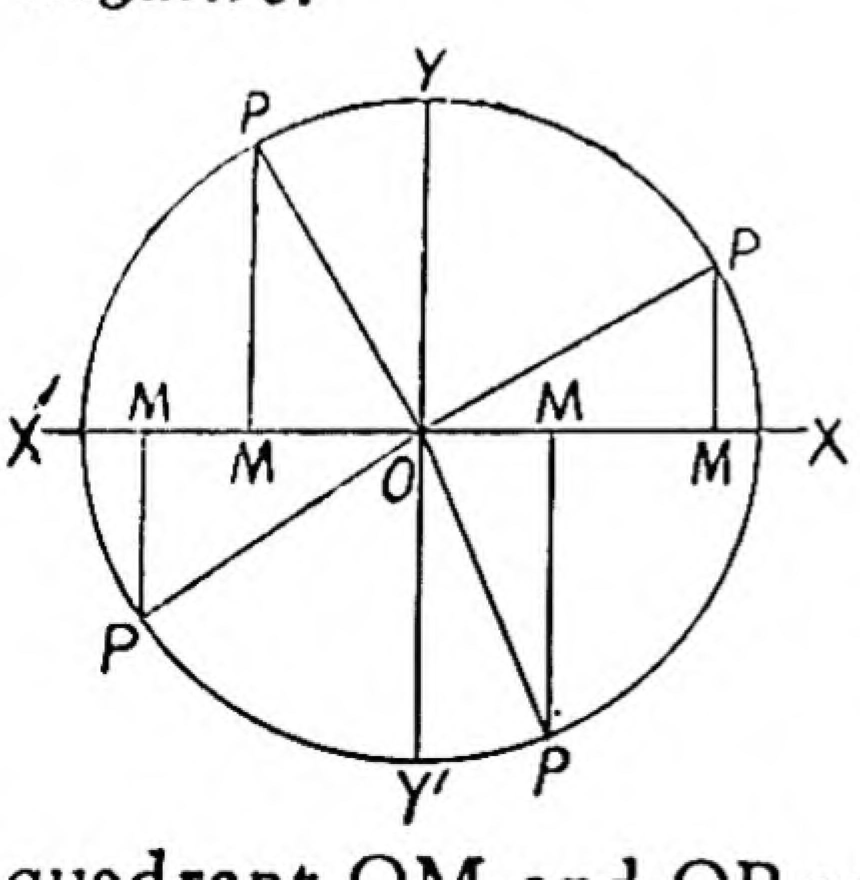
8. If $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,$ $z=r\cos\theta$, then show that $x^2+y^2+z^2=r^2$.

9. If $x=\cos\theta+\sec\theta,\ y=\sin\theta-\operatorname{cosec}\theta,$ show that $x^2+y^2=1+\sec^2\theta\operatorname{cosec}^2\theta.$

Signs of Trigonometrical Ratios.

13. **First Quadrant.** In this quadrant all the three quantities OM, MP, and OP are positive. Hence the ratios involving these quantities are positive. Thus *in the first quadrant all the six trigonometrical ratios are positive.*

Second Quadrant. In this quadrant MP and OP are positive but OM is negative. Hence the ratios involving OM are negative ; others are positive. Thus *in the second quadrant sine and cosecant are positive ; all others are negative.*



Third Quadrant. In this quadrant OP alone is positive, while OM and MP are both negative. Hence only those ratios are positive which involve both MP and OM. Thus *in the third quadrant tangent and cotangent alone are positive ; all others are negative.*

Fourth Quadrant. In this quadrant OM and OP are positive while MP is negative. Hence the ratios involving MP are negative ; others are

positive. Thus *in the fourth quadrant cosine and secant are positive ; all others are negative.*

These results can be exhibited by diagrams as follows, taking only the principal functions sin θ, cos θ and tan θ.

Sine

+	+
−	−

cosine

−	+
−	+

tangent

−	+
+	−

Sine positive **Cosine negative** **Tangent negative**	**All ratios positive**
Tangent positive **Sine negative** **Cosine negative**	**Cosine positive** **Sine negative** **Tangent negative**

14. Limits to the values of trigonometrical functions.

$\text{Sin}^2\,\theta$ and $\cos^2\theta$, both being squares are necessarily positive ; and since their sum is unity, therefore either of them can never be greater than unity. Thus $\sin^2\theta$ is never greater than unity and similarly $\cos^2\theta$ is never greater than unity.

$$\therefore\quad -1 \leqslant \sin\theta \leqslant 1 \text{ and } -1 \leqslant \cos\theta \leqslant 1.$$

i.e., sin θ as well as cos θ can never be greater than unity numerically.

Sec θ, being reciprocal of cos θ, can, therefore, be never less than unity numerically ; *i.e.*, sec θ can never lie between 1 and -1, but may have any other value. A similar remark applies to cosec θ.

From $\sec^2\theta = 1 + \tan^2\theta$, it follows that tan θ may have any value whatsoever. And cot θ, being reciprocal of tan θ, may similarly have any value.

These results could also have been arrived at from the figure of Article 10.

$\because$ MP can never be $>$ OP numerically,

$\therefore$ $\dfrac{MP}{OP}$,, ,, >1 ,,

i.e., $\sin\theta$,, ,, >1 ,,

Similarly $\cos\theta$,, >1 ,,

Again $\because$ OP ,, ,, $<$OM ,,

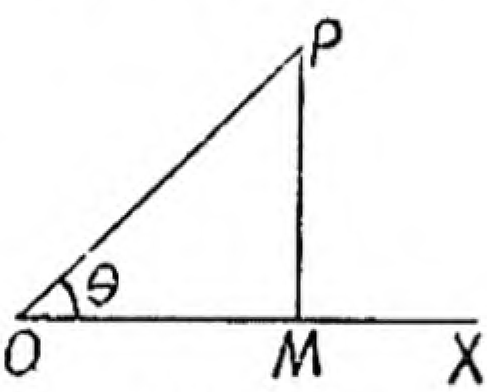
$\therefore$ $\dfrac{OP}{OM}$,, ,, <1 ,,

i.e., $\sec\theta$,, ,, <1 ,,

Similarly, $\operatorname{cosec}\theta$,, <1 ,,

Also no restrictions can be put on the ratio $\dfrac{MP}{OM}$ or $\dfrac{OM}{MP}$, therefore $\tan\theta$ and $\cot\theta$ can have any value whatsoever.

15. We are now in a position to *express the circular functions of an angle in terms of any one of them.*



First Method. Let XOP be any angle θ and let the given sine be equal to x, so that $\dfrac{MP}{OP}=x$.

Take OP=1 and therefore MP=x.

Now since the triangle MOP is always right-angled, therefore, we have

$$OM=\pm\sqrt{OP^2-MP^2}=\pm\sqrt{1-x^2}.$$

Hence $$\cos\theta=\frac{OM}{OP}=\pm\frac{\sqrt{1-x^2}}{1}=\pm\sqrt{1-\sin^2\theta}$$

$$\tan\theta=\frac{MP}{OM}=\pm\frac{x}{\sqrt{1-x^2}}=\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}};$$

$$\cot\theta=\frac{OM}{MP}=\pm\frac{\sqrt{1-x^2}}{x}=\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta};$$

$$\sec\theta=\frac{OP}{OM}=\pm\frac{1}{\sqrt{1-x^2}}=\pm\frac{1}{\sqrt{1-\sin^2\theta}};$$

$$\operatorname{cosec}\theta=\frac{OP}{MP}=\frac{1}{x}=\frac{1}{\sin}.$$

Second Method. The same results can also be obtained in another way.

Since $\sin^2\theta+\cos^2\theta=1$.

$\therefore\ \cos\theta=\pm\sqrt{1-\sin^2\theta}$;

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}};\ \cot\theta=\frac{\cos\theta}{\sin\theta}=\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\sec\theta=\frac{1}{\cos\theta}=\pm\frac{1}{\sqrt{1-\sin^2\theta}};\ \operatorname{cosec}\theta=\frac{1}{\sin\theta}.$$

Note 1.—The first method is to be preferred

Note 2.—The figure is drawn for the case when θ is an acute angle. But the same method applies to an angle of any magnitude.

Note 3.—Notice the ambiguity in sign in the first four results. When nothing is said about the magnitude of the angle, the sign of the radicals is doubtful and must be taken as $\pm$. But if the magnitude of θ be known then the radicals will not have the ambiguous sign as the proper sign of any circular function can then be determined with the help of Art. 13.

Note 4.—The first method can be expressed as follows :—

Take the ratio in terms of which the other ratios are to be expressed. Put down its value in terms of the sides of the triangle of reference, namely, OMP. Denote the numerator by x and denominator by unity. Thus the given ratio is equal to x. Then by the Pythogora's Theorem find the third side of the triangle of reference. Now put down the values of the other ratios in terms of the sides of the triangle and replace x by the ratio in terms of which the other ratios are to be expressed.

Ex. 1. Express all the circular functions of θ in terms of sec θ.

Let $\sec\theta=x=\dfrac{OP}{OM}$.

Let XOP be any angle θ and let OM=1, so that OP=x

Then $MP=\pm\sqrt{OP^2-OM^2}=\pm\sqrt{x^2-1}$.

$$\text{Hence}\quad \sin\theta=\frac{MP}{OP}=\pm\frac{\sqrt{x^2-1}}{x}=\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta};$$

$$\tan\theta=\frac{MP}{OP}=\pm\frac{\sqrt{x^2-1}}{1}=\pm\sqrt{\sec^2\theta-1};$$

$$\cot\theta=\frac{OM}{MP}=\pm\frac{1}{\sqrt{x^2-1}}=\pm\frac{1}{\sqrt{\sec^2\theta-1}}$$

$$\operatorname{cosec}\theta=\frac{OP}{MP}=\pm\frac{x}{\sqrt{x^2-1}}=\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-}};$$

$$\cos\theta=\frac{OM}{OP}=\frac{1}{x}=\frac{1}{\sec\theta}.$$

The sign which should be given to the radical can only be decided when the quadrant in which θ lies is known.

Ex. 2. Given that $\tan\theta=\frac{2}{3}$, where θ lies in the third quadrant, find the other circular functions of θ.

Here $\tan\theta=\frac{2}{3}$ and also $\tan\theta=\dfrac{MP}{OM}$. (See Fig. Art. 13 for an angle in 3rd quadrant only).

Now let MP=2 in magnitude so that OM=3 in magnitude. Hence $OP=\sqrt{2^2+3^2}=\sqrt{13}$.

Now since the angle lies in the third quadrant, therefore its sine, cosine, secant and co-secant must be negative while its tangent and co-tangent are positive.

Hence $\sin\theta=\dfrac{MP}{OP}=-\dfrac{2}{\sqrt{13}}$; $\cos\theta=\dfrac{OM}{OP}=-\dfrac{3}{\sqrt{13}}$

$$\operatorname{cosec}\theta=\frac{OP}{MP}=-\frac{\sqrt{13}}{2};\quad \sec\theta=\frac{OP}{OM}=-\frac{\sqrt{13}}{3}$$

$$\cot\theta=\frac{OM}{MP}=\frac{3}{2}.$$

Ex. 3. If $\cos A=\dfrac{3a}{2a+1}$, find the greatest and least possible values of a.

$2a\cos A+\cos A=3a$.

$$\therefore\quad a=\frac{\cos A}{3-2\cos A}=-\tfrac{1}{2}+\frac{3}{2(3-2\cos A)}.$$

Thus a is least when $3-2\cos A$ is greatest or when $\cos A=-1$. Thus least value of $a=-\frac{1}{2}+\frac{3}{10}=-\frac{1}{5}$. a is greatest when $3-2\cos A$ is least which is so when $\cos A=1$. Thus greatest value of $a=-\frac{1}{2}+\frac{3}{2}=1$.

EXERCISE V

1. Find whether the following are positive or negative:

(i) $\cos 140°$. (ii) $\sin\dfrac{2\pi}{3}$. (iii) $\tan(-122°)$.

2. Express all the circular functions of θ in terms of $\tan\theta$ when θ lies in the third quadrant.

3. Find whether θ is possible in the following :

(*i*) $\sin\theta=\frac{3}{4}$. (*ii*) $\cos\theta=\frac{6}{5}$. (*iii*) $\tan\theta=100$.

(*iv*) $\operatorname{cosec}\theta=\frac{\sqrt{3}}{2}$. (*v*) $\sin\theta=\frac{a^2+b^2}{a^2-b^2}$. (*vi*) $\sec\theta=\frac{a^2+b^2}{2ab}$.

Find the quadrant in which θ lies in the following cases :—

4. $\sin\theta=\frac{1}{3}$ and $\cos\theta=-\frac{2\sqrt{2}}{3}$.

5. $\cot\theta=-3$, $\sec\theta=\frac{\sqrt{10}}{3}$.

6. $\sec\theta=-\frac{\sqrt{13}}{3}$ and $\operatorname{cosec}\theta=-\frac{\sqrt{13}}{2}$?

7. If $\sin A=\frac{7}{25}$, find $\cos A$, A being acute.

8. If $\tan A=\frac{12}{5}$ and A is acute, find $\cos A$.

9. If $\sin\theta=\frac{1}{2}$ and θ lies in the third quadrant, find $\tan\theta$ and $\sec\theta$.

10. If $\tan\theta=3$, find $\operatorname{cosec}\theta$ when θ lies in the second quadrant.

11. If $\sin\theta=-\frac{1}{3}$, find $\tan\theta$. Explain why there are two values of $\tan\theta$.

12. If $\cos A=\frac{12}{13}$, find $\sin A$ and $\tan A$ when A lies in the fourth quadrant.

13. If $\tan\theta=3$ and θ is acute, show that

$$\frac{\sin\theta-\cos\theta}{\sec\theta-\operatorname{cosec}\theta}=0{\cdot}3.$$

14. Find $\tan A$ from the equation $3\sec^2 A+5\tan^2 A=\frac{17}{3}$ when

(*i*) A lies in the third quadrant, (*ii*) when A lies in fourth quadrant.

15. If $\sin\theta=\frac{2m}{1+m^2}$, find $\cot\theta$ and $\sec\theta$.

16. If $\tan\theta=\frac{2mn}{m^2-n^2}$, find $\sin\theta$ and $\cos\theta$.

17. ABC is a triangle having the angle ACB$=90°$. AB $=15$ feet, AC$=9$ feet; D is a point in AC such that AD$=4$ feet and CD$=5$ feet. Find (*i*) sec ABC (*ii*) cosec CBD (*iii*) cos ADB.

18. What possible values of cosec A are given by the equation $3\sec^2 A=2\operatorname{cosec} A$?

19. Is the equation $3\cos^2\theta-13\cos\theta+12=0$ possible?

20. Examine the possibility of the equation $2\cos\theta =x+\frac{1}{x}$ when x is real.

[*Hint.* The equation is $x^2-2x\cos\theta+1=0$. The discriminant of this quadratic in x is $4\cos^2\theta-4$ which must not be negative. So that $\cos^2\theta-1$ must not be negative so that $\cos\theta=1$ or >1 numerically. But $\cos\theta$ is never greater than 1. The equation is possible therefore only when $\cos\theta=1$, in which case $x=1$.]

21. Prove that the equation $\cos\theta=x+\frac{1}{x}$ is impossible if x is real.

22. Show that the equation $(a+b)^2=4ab\sin^2\theta$ is possible only when $a=b$.

23. If $\sec\theta=\frac{1}{3a}$ and $\cos\phi=\frac{b}{3}$, what are the greatest possible positive values of a and b ?

Formulae of Chapter II

A. Relations.

(*i*) $\sin\theta\times\operatorname{cosec}\theta=1$. (*ii*) $\cos\theta\times\sec\theta=1$.

(*iii*) $\tan\theta\times\cot\theta=1$. (*iv*) $\sin^2\theta+\cos^2\theta=1$.

(*v*) $\sec^2\theta=1+\tan^2\theta$. (*vi*) $\operatorname{cosec}^2\theta=1+\cot^2\theta$.

B. Signs :–

(*i*) All Trigonometrical ratios are positive in the First Quadrant.

(*ii*) Sine and cosecant are positive and others are negative in the Second Quadrant.

(*iii*) Tangent and Cotangent are positive and others are negative in Third Quadrant.

(*iv*) Cosine and Secant are positive and others are negative in Fourth Quadrant.

N.B.—Sine as well as cosine of any angle is never greater than unity numerically.

REVISION QUESTIONS II

1. Transform $(1+\cot^2\theta)\operatorname{cosec}\theta$ so that it shall contain no trigonometric functions except $\sin\theta$.

2. Express $\sin^2\theta+\cos\theta$ so that it shall contain only $\cos\theta$.

3. If $\tan\theta=\dfrac{m}{n}$, show that $\dfrac{m^2-n^2}{m^2+n^2}=\dfrac{m\sin\theta-n\cos\theta}{m\sin\theta+n\cos\theta}$.

4. An angle α lies between $180°$ and $270°$ and $\tan\alpha=\frac{24}{7}$; find the other trigonometrical ratios of α.

5. If $\tan x=2-\sqrt{3}$, find the other circular functions of x.

6. If $\operatorname{cosec}\theta-\sin\theta=a^3$ and $\sec\theta-\cos\theta=b^3$, show that $\cot\theta=\dfrac{a}{b}$.

7. If $\tan A+\sin A=m$, and $\tan A-\sin A=n$, show that $m^2-n^2=4\sqrt{mn}$.

8. If $\sec\theta=x+\dfrac{1}{4x}$, prove that $\sec\theta+\tan\theta=2x$ or $\dfrac{1}{2x}$.

9. If $\tan\theta+\sec\theta=a$, show that $(a^2+1)\sin\theta=a^2-1$.

10. Show that $(3\sin\theta-4\sin^3\theta)^2+3(\cos\theta-4\cos^3\theta)^2=1$.

11. Show that $\sec^6\theta-\tan^6\theta=1+3\tan^2\theta\sec^2\theta$.

12. Show that
$\sin^8\theta-\cos^8\theta=(\sin^2\theta-\cos^2\theta)(1-2\sin^2\theta\cos^2\theta)$.

13. Show that an angle θ can be found such that

$$\sec\theta=\frac{x^2+y^2}{2xy}.$$

14. Eliminate θ between $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$ ...(i)

and $ax\sec\theta\tan\theta+by\operatorname{cosec}\theta\cot\theta=0.$ ...(ii)

[Sol. (ii) gives $\dfrac{ax\sin\theta}{\cos^2\theta}+\dfrac{by\cos\theta}{\sin^2\theta}=0$

or $ax\sin^3\theta+by\cos^3\theta=0$

or $\dfrac{\sin^3\theta}{by}=\dfrac{\cos^3\theta}{-ax}$ or $\dfrac{\sin\theta}{(by)^{\frac{1}{3}}}=\dfrac{\cos\theta}{-(ax)^{\frac{1}{3}}}=\dfrac{1}{\sqrt{(by)^{\frac{2}{3}}+(ax)^{\frac{2}{3}}}}$

so that $\sin\theta=\dfrac{(by)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}$ and $\cos\theta=\dfrac{-(ax)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}$

Substituting these values in (i), we get

$$\frac{ax\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}{-(ax)^{\frac{1}{3}}}-\frac{by\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}{(by)^{\frac{1}{3}}}=a^2-b^2$$

or $[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}]^{\frac{3}{2}}=b^2-a^2$ or $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

15. Prove that $\sec^2\theta+\cos^2\theta$ can never be less than 2. (P. U. 1940)

[Hint. $\sec^2\theta+\cos^2\theta=(\sec\theta-\cos\theta)^2+2$ which is evidently greater than 2 except when $\sec\theta-\cos\theta=0$, in which case it becomes equal to 2].

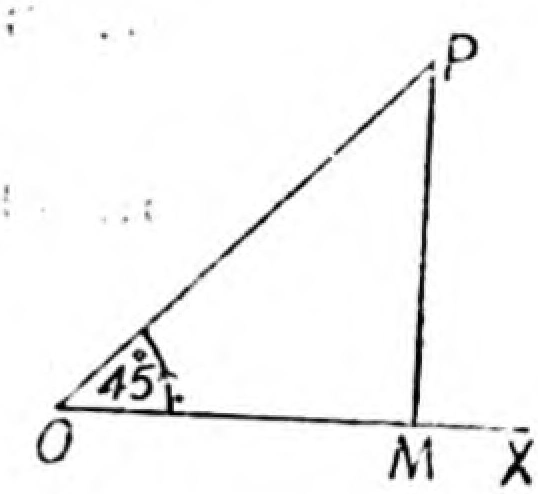
16. If $U_n=\cos^n\theta+\sin^n\theta$, show that $2U_6-3U_4+1=0$.

17. If $\sin A=\dfrac{3a}{2a+1}$ find the greatest and the least possible values of a.

CHAPTER III

TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES

16. *To find the trigonometrical functions of 45° or $\frac{\pi}{4}$.*



Let XOP be an angle of 45°. From any point P. in OP draw PM $\perp$ OX. Then $\angle$OPM=45°, because $\triangle$ OMP is right-angled.

$\therefore$ OM=MP=a (say)

$\therefore$ $OP^2=OM^2+MP^2=2a^2$

or $OP=\sqrt{2}a$.

Hence $\sin 45°=\dfrac{MP}{OP}=\dfrac{a}{\sqrt{2}a}=\dfrac{1}{\sqrt{2}}$;

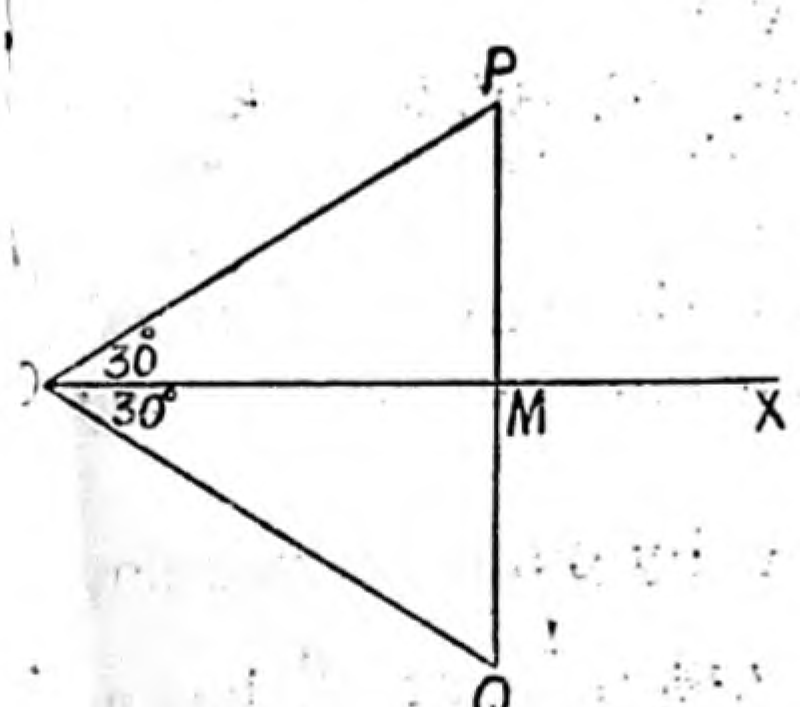
$$\cos 45°=\frac{OM}{OP}=\frac{a}{\sqrt{2}a}=\frac{1}{\sqrt{2}};$$

$$\tan 45°=\frac{MP}{OM}=\frac{a}{a}=1;\ \cot 45°=\frac{OM}{MP}=\frac{a}{a}=1;$$

$$\sec 45°=\frac{OP}{OM}=\frac{\sqrt{2}a}{a}=\sqrt{2}$$

and $\operatorname{cosec} 45°=\dfrac{OP}{MP}=\dfrac{\sqrt{2}a}{a}=\sqrt{2}$.

17. *To find the trigonometrical ratios of 30° or* $\dfrac{\pi}{6}$.



Let $\angle$XOP be 30°. Make $\angle$QOX=30° in magnitude. From any point P in OP draw PM $\perp$ OX and produce it to meet OQ in Q. Then evidently $\triangle$s MOP and MOQ are congruent.

$\therefore$ $\angle P=\angle Q=60°$, because $\angle POQ=60°$.

Hence $\triangle$POQ is equilateral.

Hence $MP=\dfrac{1}{2}PQ=\dfrac{OP}{2}=a$, (say), so that $OP=2a$

$\therefore$ $OM^2=OP^2-MP^2=4a^2-a^2=3a^2$, *i. e.*, $OM=\sqrt{3}a$.

Hence $\sin 30°=\dfrac{MP}{OP}=\dfrac{a}{2a}=\dfrac{1}{2}$;

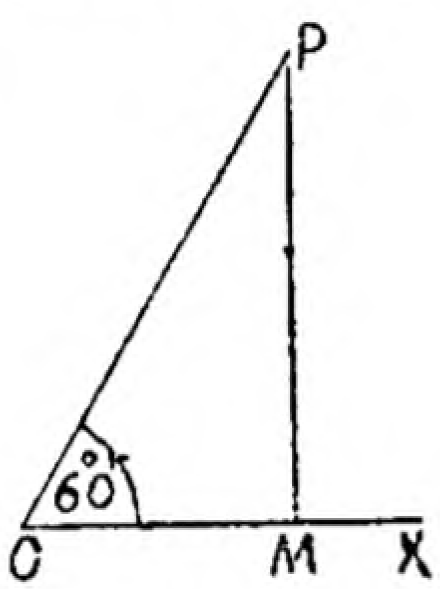
$$\cos 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2};$$

$$\tan 30^\circ = \frac{MP}{OM} = \frac{1}{\sqrt{3}}; \quad \cot 30^\circ = \frac{OM}{MP} = \sqrt{3};$$

$$\sec 30^\circ = \frac{OP}{OM} = \frac{2}{\sqrt{3}} \text{ and } \operatorname{cosec} 30^\circ = \frac{OP}{MP} = 2.$$

18. *To find the trigonometrical ratios of* 60° or $\frac{\pi}{3}$.

Let $\angle XOP = 60^\circ$. From any point P in OP draw $PM \perp OX$. Then $\angle OPM = 30^\circ$. Therefore if $OM = a$, then $OP = 2a$ and $MP = \sqrt{3}a$.



Hence $\sin 60^\circ = \frac{MP}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2};$

$$\cos 60^\circ = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2};$$

$$\tan 60^\circ = \frac{MP}{OM} = \frac{\sqrt{3}a}{a} = \sqrt{3};$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}; \ \sec 60^\circ = 2 \text{ and}$$

$$\operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$$

Note.—Angles discussed above *viz.*, 30°, 45°, 60°, lie in the first quadrant and it is for this reason that signs of the radical in the values of trigonometrical ratios are all taken positive.

19. Before proceeding to find the trigonometrical ratios of 0° and 90°, the student's attention is drawn to the following facts.

1. The division of any number by 0 has no meaning whatsoever, so that an expression like $\frac{1}{0}$ or $\frac{a}{0}$ has no meaning. The expression $\frac{1}{x}$, for example, has a defini value for all values of x, whether small or large, except tl value 0 of x.

If x decreases towards 0 (but is not allowed to ta the value 0) while it remains positive, it is obvious tl

$\frac{1}{x}$ increases and remains positive. Given any number G, however large, a value x_1 of x can be found such that $\frac{1}{x_1}>G$. For this it is sufficient to take as x_1 any number less than $\frac{1}{G}$.

It follows, therefore, that as x decreases towards 0, $\frac{1}{x}$ increases in such a way that no number, however large, can be pointed out which will not be exceeded by $\frac{1}{x}$ at same stage. This is expressed by saying that limit of $\frac{1}{x}$ when x tends to zero through positive values is infinity and is symbolically written as $\underset{x\to+0}{\mathrm{Lt}}\frac{1}{x}=\infty$.

Similarly when x tends to 0 through negative values the limit of $\frac{1}{x}$ is minus infinity or symbolically

$$\underset{x\to-0}{\mathrm{Lt}}\frac{1}{x}=-\infty.$$

The symbol ∞ means ' infinity ' but it should be noted that infinity is not a number. Such an equation as $x=\infty$ is meaningless in the ordinary sense of equality. Therefore when we say that $x\to\infty$, we shall simply mean that x is supposed to assume a succession of values which increase continually without limit.

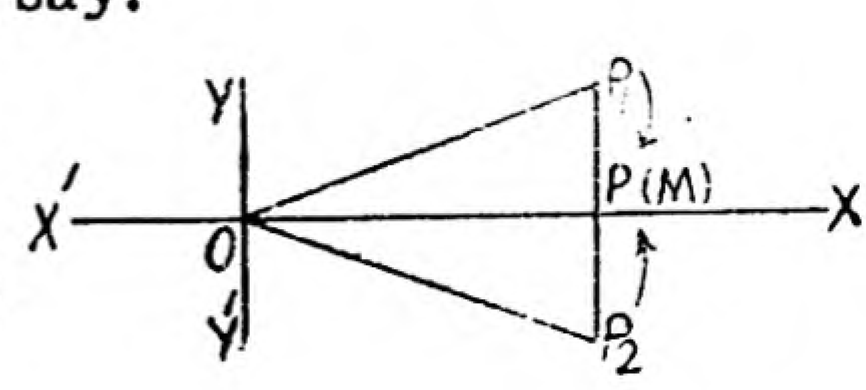
20. *To find the trigonometrical ratios of 0°.*

Let $\angle XOP$ be 0°, the revolving line OP coincides with the initial line OX. If therefore PM be supposed perpendicular to OX, P and M coincide.

$\therefore$ MP=0 and OM=OP=1, say.

Hence $\sin 0^\circ=\frac{MP}{OP}=\frac{0}{1}=0$;

$\cos 0^\circ=\frac{OM}{OP}=\frac{1}{1}=1$;



$$\sec 0° = 1,\ \tan 0° = \frac{MP}{OM} = \frac{0}{1} = 0;$$

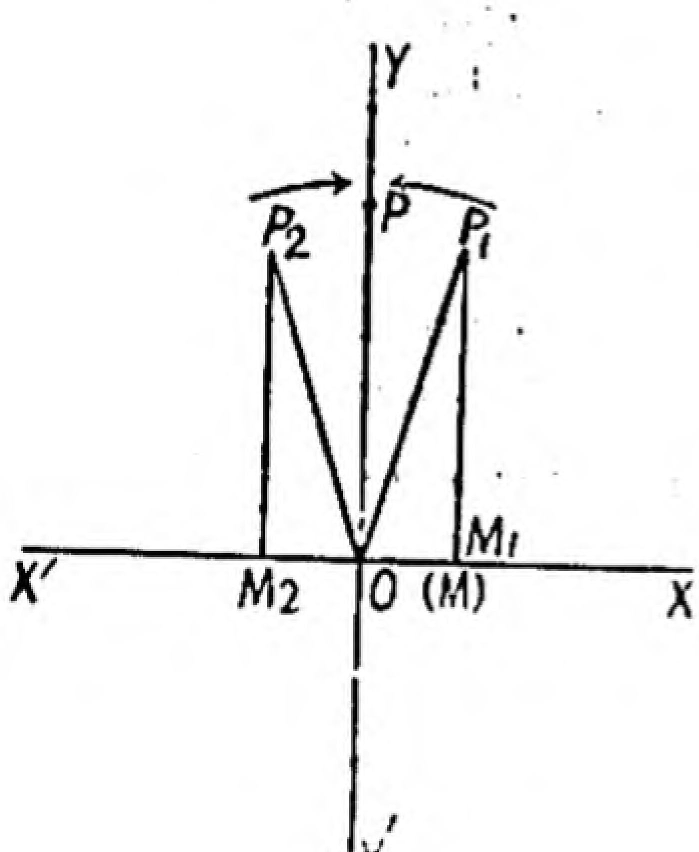
cosec 0° by definition would be equal to $\frac{OP}{OM}$ or $\frac{1}{0}$ which is meaningless. Therefore strict speaking cosec θ has no value when $\theta = 0$. But if θ is small and positive, $\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{MP}$ and is positive.

As θ tends to zero, MP also tends to zero and therefore $\frac{1}{MP}$ or cosec θ tends to infinity. Similarly if $\theta \to 0$ through negative values, then $\frac{1}{MP}$ or cosec θ tends to $-\infty$.

Hence $\underset{\theta \to 0}{\mathrm{Lt}} \operatorname{cosec}\theta = \pm\infty$

Similarly cot 0°, from definition, has no meaning. But when θ is small, $\cot\theta = \frac{OM}{MP}$. As θ tends to zero, OM tends to OP or 1 and MP tends to zero, so that $\frac{OM}{MP}$ cot θ tends to $\pm\infty$, according as θ tends to zero through positive or negative values.

21. *To find the trigonometrical ratios of 90° or* $\frac{\pi}{2}$.



Let $\angle XOP$ be 90°. From any point P in OP draw PM$\perp$OX. Then evidently M and O coincide and therefore OM=0 and MP=OP =1. (say)

Hence

$$\sin 90° = \frac{MP}{OP} = \frac{1}{1} = 1;$$

$$\cos 90° = \frac{OM}{OP} = \frac{0}{1} = 0;$$

$$\operatorname{cosec} 90° = \frac{OP}{MP} = 1;$$

$$\cot 90° = \frac{OM}{MP} = \frac{0}{1} = 0.$$

Sec 90°, by definition, would be equal to $\frac{OP}{OM}$ or $\frac{1}{0}$, which is meaningless. But if θ is less than 90°, sec θ $=\frac{OP}{OM}=\frac{1}{OM}$ and is positive, because OM is positive. As θ tends to 90°, OM tends to zero and therefore $\frac{1}{OM}$ or sec θ tends to $+\infty$. Similarly if θ is greater than 90°, then sec $\theta=\frac{OP}{OM}=\frac{1}{OM}$ and is negative, because OM is negative: and as θ tends to 90°, OM tends to zero so that sec θ tends to $-\infty$. Hence

$\underset{\theta\to 90^\circ}{\text{Lt}}$ sec $\theta=\pm\infty$ according to $\theta\to 90^\circ$ through values less than 90° or greater than 90°.

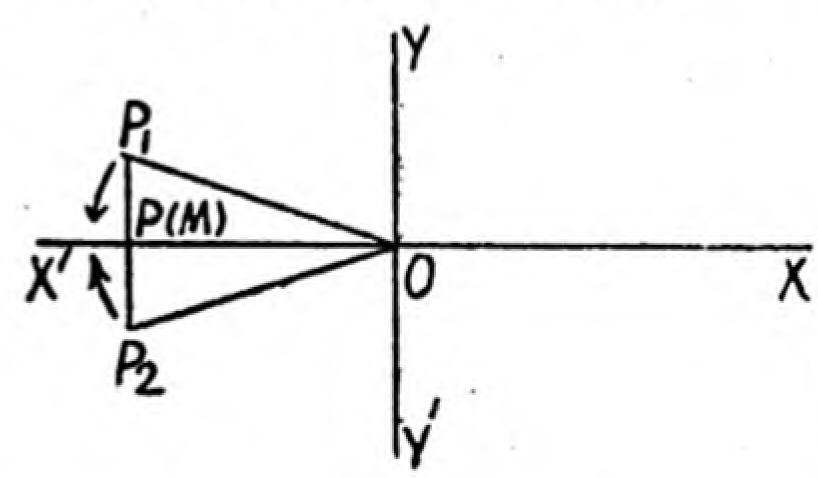
Similarly it follows that

$\underset{\theta\to 90^\circ}{\text{Lt}}$ tan $\theta=\pm\infty$, according as $\theta\to 90^\circ$ through values less than or greater than 90°.

22. *To find the trigonometrical ratios of* 180° *or* π.

Let $\angle$XOP be 180°. The revolving line OP coincides with OX′. If, therefore, PM be supposed perpendicular to OX′: P and M coincide.

$\therefore$ MP=O. If OP=1, OM=−1.



$$\text{Hence sin } 180^\circ=\frac{MP}{OP}=\frac{0}{1}=0;\ \cos 180^\circ=\frac{OM}{OP}=\frac{-1}{1}=-1;$$

$$\tan 180^\circ=\frac{MP}{OM}=\frac{0}{OM}=0;\ \sec 180^\circ=\frac{OP}{OM}=\frac{1}{-1}=-1.$$

Cosec 180° by definition would be equal to $\frac{OP}{MP}$ or $\frac{1}{0}$, which is meaningless. Therefore strictly speaking cosec θ has no value when $\theta=180^\circ$. But if θ is slightly less than 180° and is therefore in the second quadrant,

$\text{cosec}\,\theta = \frac{OP}{MP} = \frac{1}{MP}$ and is positive.

As θ tends to $180°$, MP tends to zero and therefore $\frac{1}{MP}$ or cosec θ tends to infinity. Similarly if $\theta \to 180°$ through values greater than $180°$, then $\frac{1}{MP}$ or cosec θ tends to $-\infty$.

Hence $\underset{\theta \to 180°}{\text{Lt}} \text{cosec}\,\theta = \pm\infty$

Similarly $\underset{\theta \to 180°}{\text{Lt}} \cot\theta = \mp\infty$, according as $\theta \to 180°$ through values less than or greater than $180°$.

23. *To find the trigonometrical ratios of* $270°$ *or* $\frac{3\pi}{2}$.

Let $\angle XOP$ be $\frac{3\pi}{2}$; the revolving line OP coincides with OY′. If therefore, PM be supposed perpendicular to OX, O and M coincide and therefore, $OM = 0$. If $OP = 1$, then $MP = -1$.

Hence $\sin 270° = \frac{MP}{OP} = \frac{-1}{1} = -1$; $\cos 270° = \frac{OM}{OP} = \frac{0}{1} = 0$

$\cot 270° = \frac{OM}{MP} = \frac{0}{-1} = 0$; $\text{cosec}\,270° = \frac{OP}{MP} = \frac{1}{-1} = -1.$

Since OM tends to 0 and MP tends to -1,

$\underset{\theta \to 270}{\text{Lt}} \tan\theta = \pm\infty$, according as θ is less than or greater than $270°$.

Similarly $\underset{\theta \to 270°}{\text{Lt}} \sec\theta = \mp\infty$.

24. *To find the trigonometrical ratios of* $360°$ *or* 2π,

When the revolving line OP, starting from its initial position OX, has turned through an angle of $360°$ it coincides with OX. Hence the trigonometrical ratios of $360°$ are the same as those of $0°$.

i.e., $\sin 360° = 0$; $\cos 360° = 1$; $\tan 360° = 0$; $\cot 360° = \mp\infty$; $\sec 360° = 1$; $\text{cosec}\,360° = \pm\infty$.

Ex. 1. Solve the equations $2\cos^2\theta+7\sin\theta-5=0$.

The equation can be written as,

$$2(1-\sin^2\theta)+7\sin\theta-5=0$$

or $2-2\sin^2\theta+7\sin\theta-5=0$ or $2\sin^2\theta-7\sin\theta+3=0$

$$\therefore \sin\theta=\frac{7\pm\sqrt{49-24}}{4}=\frac{7\pm5}{4}=3 \text{ or } \frac{1}{2}.$$

Now $\sin\theta$ is never greater than 1 and therefore the value 3 is rejected. Hence we get only

$$\sin\theta=\frac{1}{2}, \text{ which gives us } \theta=\frac{\pi}{6}.$$

This angle $\frac{\pi}{6}$ is said to be inverse sine of $\frac{1}{2}$ and is written as $\sin^{-1}\frac{1}{2}$.

Ex. 2. What is $\sin^{-1}\frac{\sqrt{3}}{2}$?

It is an angle whose sine is $\frac{\sqrt{3}}{2}$, *i.e.*, 60°.

Ex. 3. Show that

$$4\cos^3\frac{\pi}{6}-3\cos\frac{\pi}{6}+1=3\cos\frac{\pi}{3}-4\cos^3\frac{\pi}{3}.$$

$$\text{L. H. S.}=4.\left(\frac{\sqrt{3}}{2}\right)^3-3.\frac{\sqrt{3}}{2}+1=4.\frac{3\sqrt{3}}{8}-\frac{3\sqrt{3}}{2}+1=1.$$

$$\text{R.H.S.}=3\times\tfrac{1}{2}-4(\tfrac{1}{2})^3=\tfrac{3}{2}-\tfrac{1}{2}=1.$$

Ex. 4. Solve for R and θ,

$$R\sin\theta=1 \qquad \ldots\ldots(i)$$

$$R\cos\theta=\sqrt{3} \qquad \ldots\ldots(ii)$$

given that θ is acute and positive.

Squaring and adding (*i*) and (*ii*), we get

$R^2(\sin^2\theta+\cos^2\theta)=4$ or $R^2=4$

$$\therefore \quad R=\pm2.$$

The value -2 is rejected because this leads to the equation $\sin\theta=-\frac{1}{2}$ which is not possible since θ is acute and positive $\therefore$ $R=2$.

When $R=2$, $\sin\theta=\frac{1}{2}$ and therefore $\theta=30°$.

EXERCISE VI

Prove that

1. $3 \tan 60^\circ = \tan^3 60^\circ$.
2. $3 \cot^2 60^\circ - \sin^2 45^\circ - \cos 60^\circ = 0$.
3. $2 \sin^2 45^\circ - 6 \tan^2 30^\circ + \operatorname{cosec} 30^\circ = 1$.
4. $\operatorname{cosec}^2 45^\circ + 2 \sec^2 30^\circ - 8 \cot^2 60^\circ = 2$.
5. (*i*) $\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ = \cos 30^\circ$.
   (*ii*) $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ = 0$.

If $A = 30^\circ$, verify that

6. $\cos 2A = \cos^2 A - \sin^2 A$.
7. $\sin 3A = 3 \sin A - 4 \sin^3 A$.
8. $\cos 3A = 4 \cos^3 A - 3 \cos A$.

If $A = 60^\circ$, $B = 30^\circ$, verify that

9. $\sin (A+B) = \sin A \cos B + \cos A \sin B$.
10. $\cos (A-B) = \cos A \cos B + \sin A \sin B$.
11. Is the relation $\sin 2\theta = 2 \sin \theta \cos \theta$ true when $\theta = 30^\circ$ or 45°?

Solve the following equations :

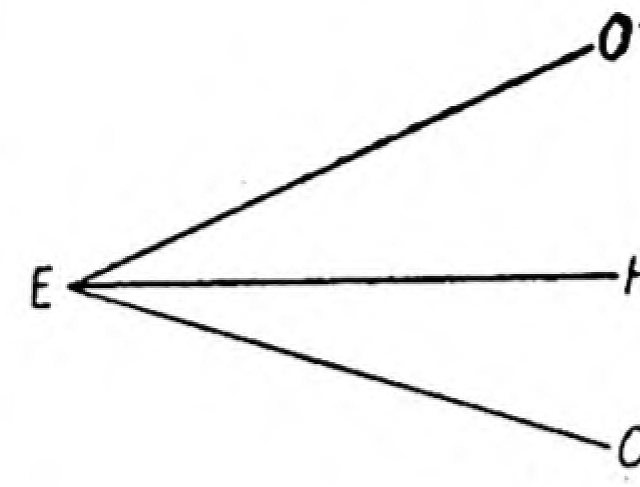
12. $x \cot^2 45^\circ \sec^2 60^\circ = 12 \sin^2 90^\circ$.
13. $2 \sin \theta = \tan \theta$.
14. $2 \cos^2\theta - 1 = 1 - \sin^2\theta$.
15. $\tan \theta \sin \theta - \sin\theta = 0$.
16. $\tan \theta + \cot \theta = 2$.
17. Give that $\sin (A-B) = \frac{1}{2}$ and $\cos (A+B) = \frac{1}{2}$, find A and B.
18. Given that $\tan (A-B) = \frac{1}{\sqrt{3}}$ and $\cos (A+B) = 0$, find A and B.
19. When $A = 45^\circ$, $B = 30^\circ$ and $C = 60^\circ$, find the value of $\sin A \cos B \cos C - \cos A \sin B \sin C + \cos A \cos B \cos C$.

Heights and Distances

25. One of the applications of elementary trigonometry is the finding of heights and distances from the knowledge of some known angles, heights and distances. Thus trigonometry is highly useful in land survey. Also with the help of trigonometry we can measure the heights or the distances of points which are otherwise inaccessible; for example, the distances of the sun, the moon, and the planets. The method will be best illustrated by the examples that follow.

Definition.

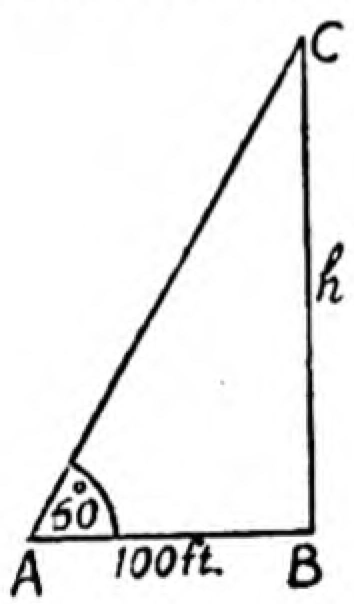
EH is the horizontal line, E being the observer and O and O′ are any two objects in the vertical plane containing EH. Then $\angle HEO$ is defined as the **Angle of Elevation** of O, and is sometimes called **altitude** of O.



$\angle HEO'$ is called the **Angle of Depression** of O′.

Ex. 1. A tower stands on a horizontal plane. A man on the ground 100 feet from the foot of the tower finds that the angle of elevation of the top is 60°. Find the height of the tower.

Let BC be the tower of height h and let **A** be the position of the observer.



Then, $AB=100$ feet and $\angle BAC=60°$.

Also $\angle ABC=90°$.

Hence $\tan 60°=\dfrac{BC}{AB}=\dfrac{h}{100}$

$$\therefore \quad h=100 \tan 60°$$
$$=100\sqrt{3}$$
$$=173{\cdot}2.$$

Therefore the height of the tower is 1732 ft.

Ex. 2. A person standing on the bank of a river finds that the angle of elevation of the top of a tree on the opposite bank is 45° ; on going back 50 yards, he finds that the angle of elevation is 30° ; find the height of the tree and the breadth of the river.

Let x be the height of the tree.

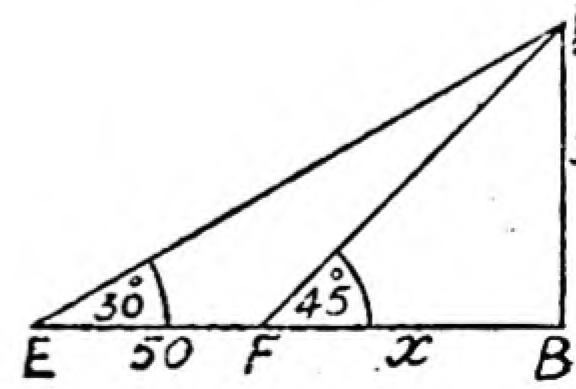
Since $\angle DFB=45°$, $\therefore$ $\angle BDF=45°$.

Hence $FB=BD=x$.

Now $\tan 30°=\dfrac{BD}{EB}=\dfrac{x}{50+x}$,

i.e., $\dfrac{1}{\sqrt{3}}=\dfrac{x}{x+50}$,

or $50+x=\sqrt{3}x$.



$$\therefore \quad x=\frac{50}{\sqrt{3}-1}=\frac{(50\ \sqrt{3}+1)}{2}$$
$$=25\times(2{\cdot}732),\text{ approximately}=68{\cdot}3.$$

Therefore the height of the tree = breadth of the river = 68·3 yds. approximately.

Ex. 3. An aeroplane is observed at the same time by two anti-aircraft batteries distant 6000 feet apart to be at elevations of 30° and 45° respectively. Assuming that the aeroplane is travelling directly towards the two batteries, find its height and its horizontal distance from the nearer battery.

Let A and B be the two batteries and C the aeroplane,

Draw CD $\perp$ to AB and let CD, the height of the aeroplane be x and let BD $=y$, so that AD $=6000-y$.

Then $\angle$CBD $=45^\circ$ and since $\tan 45^\circ=1$,

we have $1=\dfrac{x}{y}$ or $y=x$.

Also $\angle$CAD $=30^\circ$ and since $\tan 30^\circ=\dfrac{1}{\sqrt{3}}$,

we have $\dfrac{1}{\sqrt{3}}=\dfrac{x}{6000-y}=\dfrac{y}{6000-y}$ since $x=y$.

Hence $6000-y=\sqrt{3}y$ or $y=\dfrac{6000}{\sqrt{3}+1}=21696$ feet.

Ex. 4. A man standing a feet behind and opposite the middle of a football goal observes that the angle of elevation of the nearer cross bar is α and that the angle of elevation of the farther cross bar is β. Show that the length of the field is $a\ \dfrac{\tan\alpha-\tan\beta}{\tan\beta}$.

Let O be the observer and let AB and CD be the heights of the two cross bars. OAC being a st. line OA $=a$ ft. and $\angle$BOA $=\alpha$ and $\angle$DOC $=\beta$.

As $\dfrac{AB}{OA}=\tan\alpha$, $\therefore$ AB $=a\tan\alpha$. Also as AB $=$ CD.

$\therefore$ CD $=a\tan\alpha$.

Again, $\dfrac{OC}{CD}=\cot\beta$, $\therefore$ OC $=$ CD $\cot\beta=a\tan\alpha\cot\beta$.

Hence length AB of the field is given by

$$AB = OC - OA = a \tan\alpha \cot\beta - a$$

$$= a\left(\frac{\tan\alpha}{\tan\beta} - 1\right)$$

$$= a\,\frac{(\tan\alpha - \tan\beta)}{\tan\beta}.$$

EXERCISE VII

1. From a point 80 feet from the foot of a tower the angle of elevation of the top is 30° ; find the height of the tower.

2. At a point 200 feet from a tower which stands on a horizontal plane, the angle of elevation of the top is 60°. Find its height ; also find at what distance the angle of elevation will be 30°.

3. A kite string is 350 yards long and its angle of elevation is 60°. Find the height of the kite above the ground.

4. From the top of a cliff 125 feet high a man observes the angle of depression of a boat to be 30°. Find the distance of the boat from the foot of the cliff.

5. A vertical post casts a shadow 20 ft. long when the altitude of the sun is 60°. Find the length of the shadow when the altitude of the sun is 30°.

6. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the foot to the top of the stump is 40 feet. What was the height of the tree ?

7. A person standing on the bank of a river finds that the elevation of the top of a tower on the opposite bank is 60° ; on going back 25 feet he finds that the elevation of the top is 35°. Find the breadth of the river between the man and the tower.

30°

8. From the top of a tower 100 feet high the angles of depression of the top and the bottom of a house are 30° and 45°. Find the height of the house and its distance from the tower.

9. From the top of a tower 100 feet high the angles of

depression of two objects situated on the plane on which the tower stands, due north of the tower, are 60° and 45°. Find the distance between the objects.

10. A straight tunnel AB is bored horizontally through a mountain. The distance over the mountain ACB is 5 miles and the sides of the mountain slope at angles of 30° and 45°. Find the length of the tunnel and the height of the top C above AB.

11. An aircraft is observed at the same instant from two places A, B 10 miles apart, at elevations of 30° and 60°, being then vertically above some point between A and B. Half a minute later it is vertically above B. Find its height and its speed.

12. A telegraph pole of diameter 1 ft. 8 in. is strengthened by a wire cable, which passes once round the pole at a height of 10 ft. from the ground. Find the length of the cable required, given that the cable is inclined at an angle of 50° to the ground.

26. Above we have calculated trigonometrical ratios for the angles 30°, 45°, 60°, 90°, 0°. In a subsequent chapter we shall give methods to calculate the values of trigonometrical ratios of one or two angles more. But towards the end of the book are given tables of values under the headings 'Natural Sines,' 'Natural Tangents', etc., in which are tabulated the values of trigonometrical ratios for all angles from 0° to 90° correct to four places of decimals. The method of using these tables is illustrated below :—

Firstly. *To write down the trigonometrical ratios of an angle* ; *e.g.*, to find sin 30° 18′ and sin 30° 21′.

Turn to the pages in which the heading is *Natural Sines* ; run down the first column under degrees till 30° is reached and then look along the row of 30° till the minute column under 18′ is reached : we get the number 5045 there.

Thus sin 30° 18′ = ·5045.

Now to find sin 30° 21′, we proceed as in the former case and get sin 30° 18′ and then look at the columns on the right hand under the heading *mean differences.* Run down the column under 3′ till you get to the row 30° ; you find as the difference for 3′, Therefore it means :

sin 30° 18′ = 5045 (as before)
Diff. for 3′ = ·0008 (now found)

∴ sin 30° 21′ = ·5053.

Note. For cosines of angles refer to pages with heading *Natural cosines* and for tangents to pages with heading *Natural Tangents* and for cotangents to pages with heading *Natural Cotangents*.

Secondly. *To find the angles lying between 0° and 90° corresponding to a given trigonometric ratio, e. g.,* given $\tan\theta = 1{\cdot}1231$, to find θ.

Turn to the pages of *natural tangents* and try to find out the number nearest to 1·1231 and less than it. We find that the nearest number is 1·1224 given in the row of 48° and in the column of 18′; this means that tan 48° 18′ = 1·1224. Now the difference between 1·1231 (the given tangent) and 1·1234 is ·0007, *i. e.*, 7 (omitting the point and the zeros). Examine the difference column along the row of 48° and find the number 7 given under the difference 1′. This means that the angle found first, *viz.*, 48° 18′ must be increased by 1′, *i.e.*, $\theta = 48°\ 19'$,

Similar process is to be observed for other cases.

It must be observed that the mean difference is to be added in the cases of sine and tangent, and subtracted in the case of cosine and cotangent for an increment in the angle.

Ex. Solve $3\sec^2\theta = 8\tan\theta - 2$.

Since $\sec^2\theta = 1 + \tan^2\theta$,

we have $3(1+\tan^2\theta) = 8\tan\theta - 2$.

or $3\tan^2\theta - 8\tan\theta + 5 = 0$

i. e., $(\tan\theta - 1)(3\tan\theta - 5) = 0$

∴ either $\tan\theta = 1$ which gives $\theta = 45°$

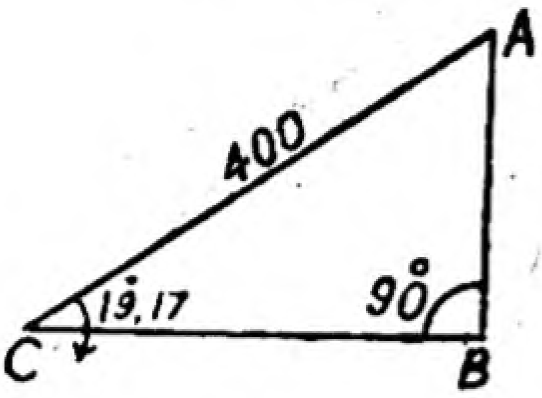
or $\tan\theta = \frac{5}{3} = 1{\cdot}6667$, which gives

$\theta = 59°\ 3'$ (From tables of Natural Tangents).

Note.—Whenever we omit a figure in the 5th place of decimals, we add 1 to the figure in the fourth place if the omitted figure be 5 or a number greater than 5.

Ex. 1. The side of a rt. angled triangle opposite to rt. angle is 400 ft. and one angle is 19° 17′. Find the other two sides.

Here let ABC be the given triangle rt. angled at B.



Then $AC = 400'$ ft.

$\angle ACB = 19° 17'$

$\therefore \quad AB = AC \sin ACB$
$= 400 \sin 19° 17'$
$= 400 \times \cdot 3303$ ft.
$= 132 \cdot 12$ ft.

$CB = AC \cos ACB$
$= 400 \cos 19° 17'$
$= 400 \times \cdot 9439$
$= 377 \cdot 56$ ft.

Ex. 2. From a point 50 ft. from the foot of a tree the angle of elevation of the top is 70°; find the height of the tree.

Here $AB = h$

$= BC \tan 70° = 50 \times 2 \cdot 7475$ ft. $= 137 \cdot 375$ ft.

EXERCISE VIII

1. If the sine of an angle be ·2116, find the angle.
2. If the cosine of an angle be ·9731, find the angle.
3. If the tangent of an angle be ·3669, find the angle.

Solve the following triangles, using four figure tables :—

4. $A = 34°$, $B = 56°$, $c = 10$. 5. $B = 35°$, $C = 90°$, $b = 5$.
6. $A. = 80°$, $C = 90°$. $b = 435$.
7. $C = 90°$, $a = 500$, $A = 50° 17'$.
8. $a = 50$, $B = 75°$, $C = 90°$.

9. Each leg of a step ladder is 8 ft. long and it stands on level ground with its feet 5 ft. apart. Find the angle which each leg makes with the ground.

10. The two tangents from a point P to a circle of radi us 5″ are inclined at an angle of 44° to each other. Find the length of either tangent.

Formulae of Chapter III

(An Aid to Memory)

Angle	0°	30°	45°	60°	90°
Sine	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
Cosine	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
Tangent	$\sqrt{\frac{0}{4-0}}$	$\sqrt{\frac{1}{4-1}}$	$\sqrt{\frac{2}{4-2}}$	$\sqrt{\frac{3}{4-3}}$	$\sqrt{\frac{4}{4-4}}$

Note 1.—Only first two may be memorised because by dividing the sine of the angle by its cosine, tangent can be obtained and cotangent, secant and cosecant are reciprocals of tangent, cosine and sine respectively.

Note 2.—The values of Trigonometrical ratios of angles bigger than 90° will form the subject matter of the next Chapter.

REVISION QUESTIONS III

1. Verify by taking $A=60°$ and $B=30°$ that
 (*i*) $\cos(A-B)$ is not equal to $\cos A-\cos B$.
 (*ii*) $\sin(A+B)$ is not equal to $\sin A+\sin B$.

2. Verify by taking $A=30°$ that

 (*i*) $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$. (*ii*) $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$.

3. Verify by taking $A=30°$ that
 (*i*) $\sin 2A$ is not equal to $2\sin A$.
 (*ii*) $\cos 2A$ is not equal to $2\cos A$.

4. Solve for θ, $2\sin^2\theta-3\sin\theta+1=0$ when θ lies in the first quadrant.

5. From the top of a cliff, 254 ft. high, the angle of depression of a ship was found to be 9°, and that of the edge of the sea 72°; how far distant was the ship from the edge of the sea?

6. The angle of elevation of a cloud from a point 400 feet above a lake is 30° and the angle of depression of its reflection in the lake is 60°. Find the height of the cloud.

7. A person on the bank of a river observes that the straight line between himself and a particular point on the opposite bank makes an angle of $60°$ with the stream. After walking along the bank down-stream a distance of 150 feet the angle is $30°$. Find the width of the river. (P. U.)

8. From a light-house the angles of depression of two ships on opposite sides of the light-house are observed to be $30°$ and $45°$. If the height of the light-house be 300 feet, find the distance between the ships if the line joining them passes through the foot of the light-house. (P. U. 1941)

9. An observer in a boat is being rowed away from a cliff 150 feet high and it takes 2 minutes for the angle of elevation of the top of the cliff to change from $60°$ to $45°$. Find the speed of the boat.

10. A flagstaff 20 feet high stands on the top of a cliff and from a point on a level with the base of the cliff the angles of elevation of the top and the bottom of the flagstaff are found to be $45°$ and $30°$. Find the height of the cliff.

11. There are two towers on a horizontal plane. Observed from the foot of the first the angle of elevation of the top of the other is $60°$; when observed from the foot of the second, the angle of elevation of the top of the first is $30°$. Prove that the second tower is three times as high as the first.

12. From a point on the ground on one side of a street 40 feet wide, the height of a house on the opposite side is observed to subtend an angle of $60°$ and the top of a window is at an angular altitude of $45°$. Determine the height of the house and the distance of the top of the window from the ground.

13. In a quadrilateral ABCD, CD=6 in., AD=8 in., angles at C and D are right angles and angle at A is $140°$. Find the lengths of the sides AB and BC.

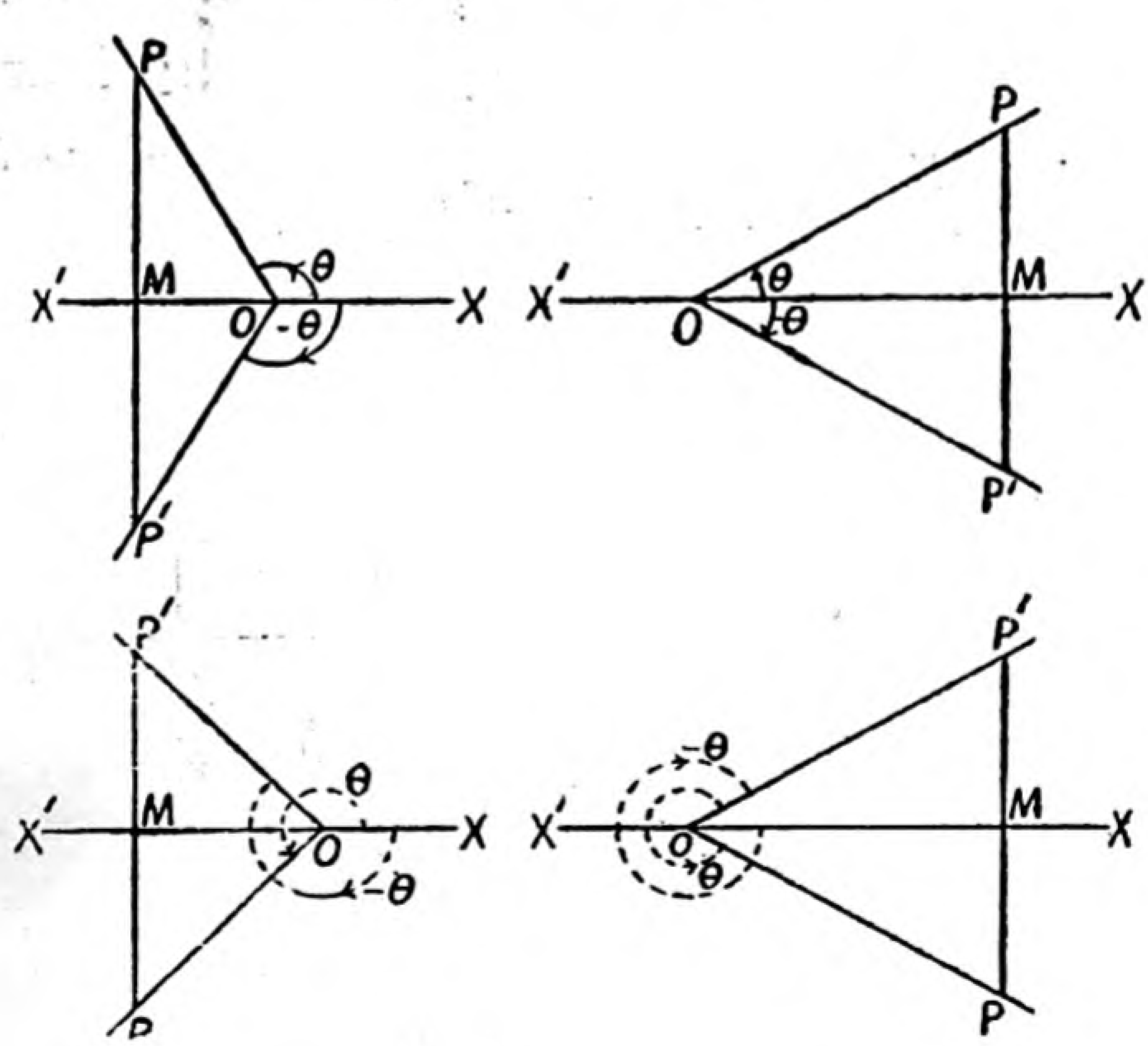
14. A garage with a span roof has a rectangular floor of 10 ft. wide and 16 ft. long and the pitch of the roof is $30°$. Find the area of the roof.

CHAPTER IV

TRIGONOMETRICAL RATIOS OF ALLIED ANGLES

27. *To compare the trigonometrical ratios of the angles* $+\theta$ *and* $-\theta$.

Let the revolving line starting from the position OX describe the angle XOP $(=+\theta)$ and the angle XOP′ $(=-\theta)$. From any point P in OP draw PM $\perp$ OX or OX′, and produce it to cut OP in P′. Then evidently the triangles OPM and OP′M are congruent. Therefore, having regard to the signs of lines, we have



OP=OP′ ; MP′=−MP ; OM=OM.

$$\therefore \quad \sin(-\theta)=\frac{MP'}{OP}=-\frac{MP}{OP}=-\sin\theta\,;$$

$$\cos(-\theta)=\frac{OM}{OP'}=\frac{OM}{OP}=\cos\theta\,;$$

$$\tan(-\theta)=\frac{MP'}{OM}=-\frac{MP}{OM}=-\tan\theta\,;$$

$$\cot(-\theta)=\frac{OM}{MP'}=-\frac{OM}{MP}=-\cot\theta\,;$$

$$\sec(-\theta)=\frac{OP'}{OM}=\frac{OP}{OM}=\sec\theta\,;$$

and $\operatorname{cosec}(-\theta)=\dfrac{OP'}{MP'}=-\dfrac{OP}{MP}=-\operatorname{cosec}\theta.$

Ex. With the help of this article, find the trigonometrical ratios of 0°.

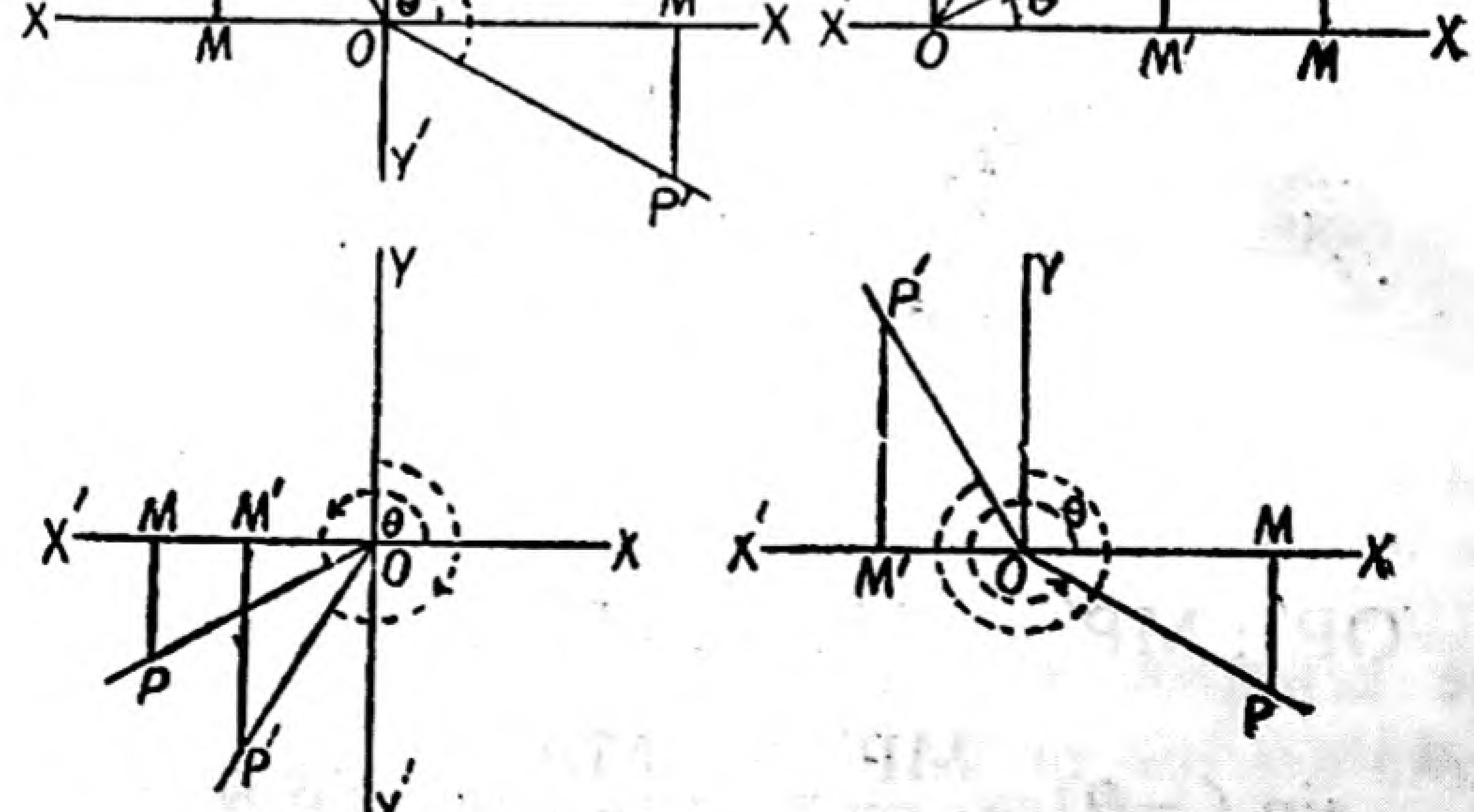
We have $\sin 0^\circ=\sin(-0^\circ)$

$=-\sin 0^\circ \ (\because -0=+0).$

By transposition, $2\sin 0^\circ=0$; *i.e.*, $\sin 0^\circ=0$.

The remaining ratios can now be easily found.

28. *To compare the trigonometrical ratios of* θ *and* $(90^\circ-\theta)$; *i e., of complimentary angles.*



Let the revolving line, starting from OX, trace out the angle XOP equal to θ; then let the revolving line coincide with OY and then revolve in the opposite direction through θ so that it has described an angle XOP′ equal to $90° - \theta$.

From any point P in OP, draw PM perpendicular to OX or OX produced. Cut off OP′=OP from OP′. Draw P′M′ perpendicular to OX or OX′. Then the triangles OPM and P′M′O are evidently congruent. Therefore having regard to signs of lines we have

$$OP' = OP;\ OM' = MP \text{ and } M'P' = OM,$$

$$\therefore \quad \sin(90° - \theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta;$$

$$\cos(90° - \theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta;$$

$$\tan(90° - \theta) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta;$$

$$\cot(90° - \theta) = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta;$$

$$\sec(90° - \theta) = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta;$$

$$\text{and } \operatorname{cosec}(90° - \theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

Ex. With the help of this article find the trigonometrical ratios of 45°.

We have $\sin 45° = \cos(90° - 45°) = \cos 45°$

Dividing by cos 45°, we have $\tan 45° = 1$.

The remaining ratios can now be easily found.

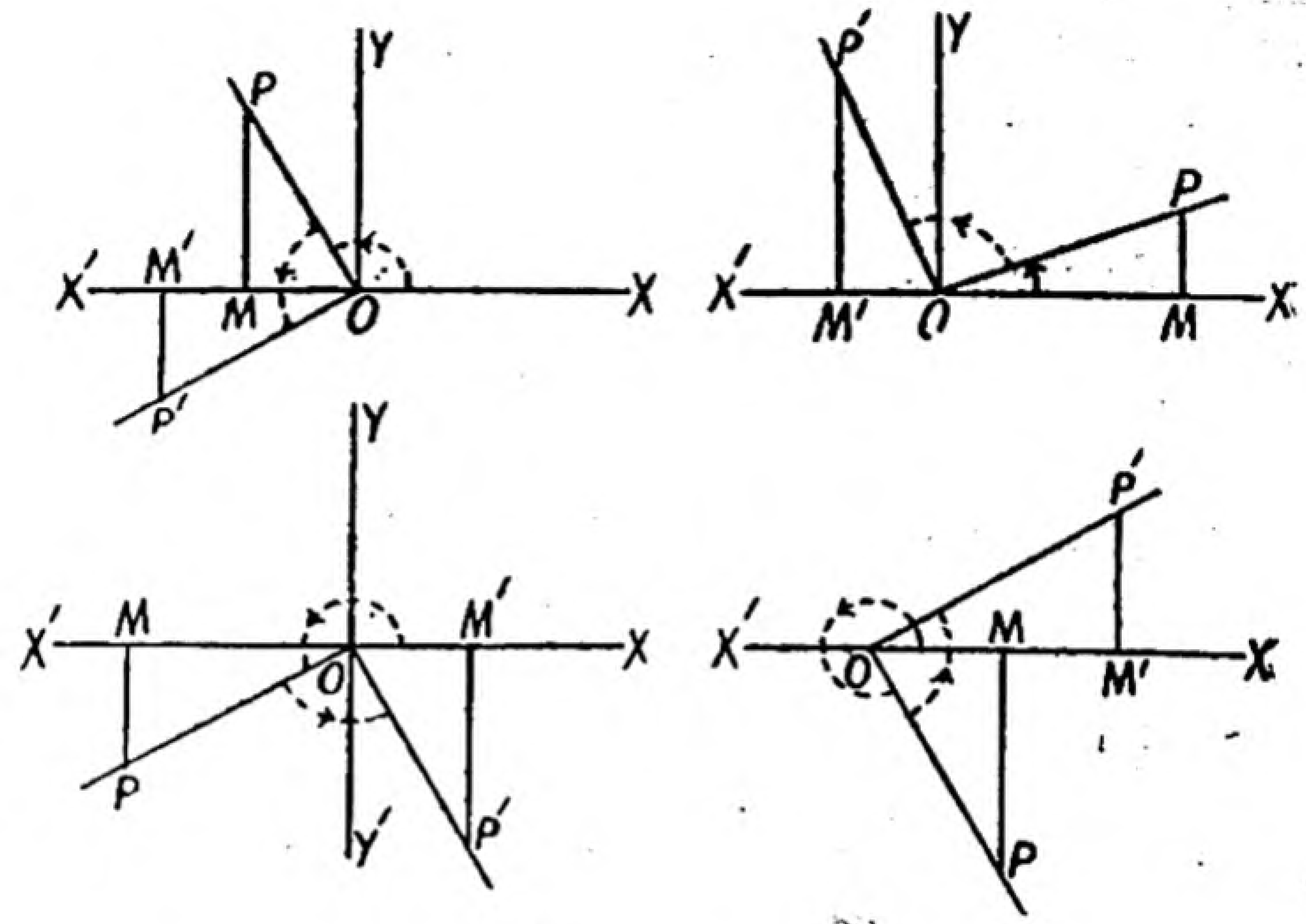
29. *To compare the trigonometrical ratios of θ and $90° + \theta$.*

Let the revolving line OP starting from OX revolve and describe the angle XOP equal to θ; let it revolve further from the position OP through a right angle in the positive direction to the position OP′.

From any point P in OP draw PM $\perp$ OX or OX′. From OP′ cut off OP′=OP. Draw P′M′ $\perp$ OX or OX′.

Then the triangles OMP and P′M′O are evidently congruent. Therefore having regard to the signs of lines, we have

OP′ = OP ; M′P′ = OM and OM′ = −MP.



$$\therefore \quad \sin(90°+\theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta\,;$$

$$\cos(90°+\theta) = \frac{OM'}{OP'} = -\frac{MP}{OP} = -\sin\theta\,;$$

$$\tan(90°+\theta) = \frac{M'P'}{OM'} = -\frac{OM}{MP} = -\cot\theta\,;$$

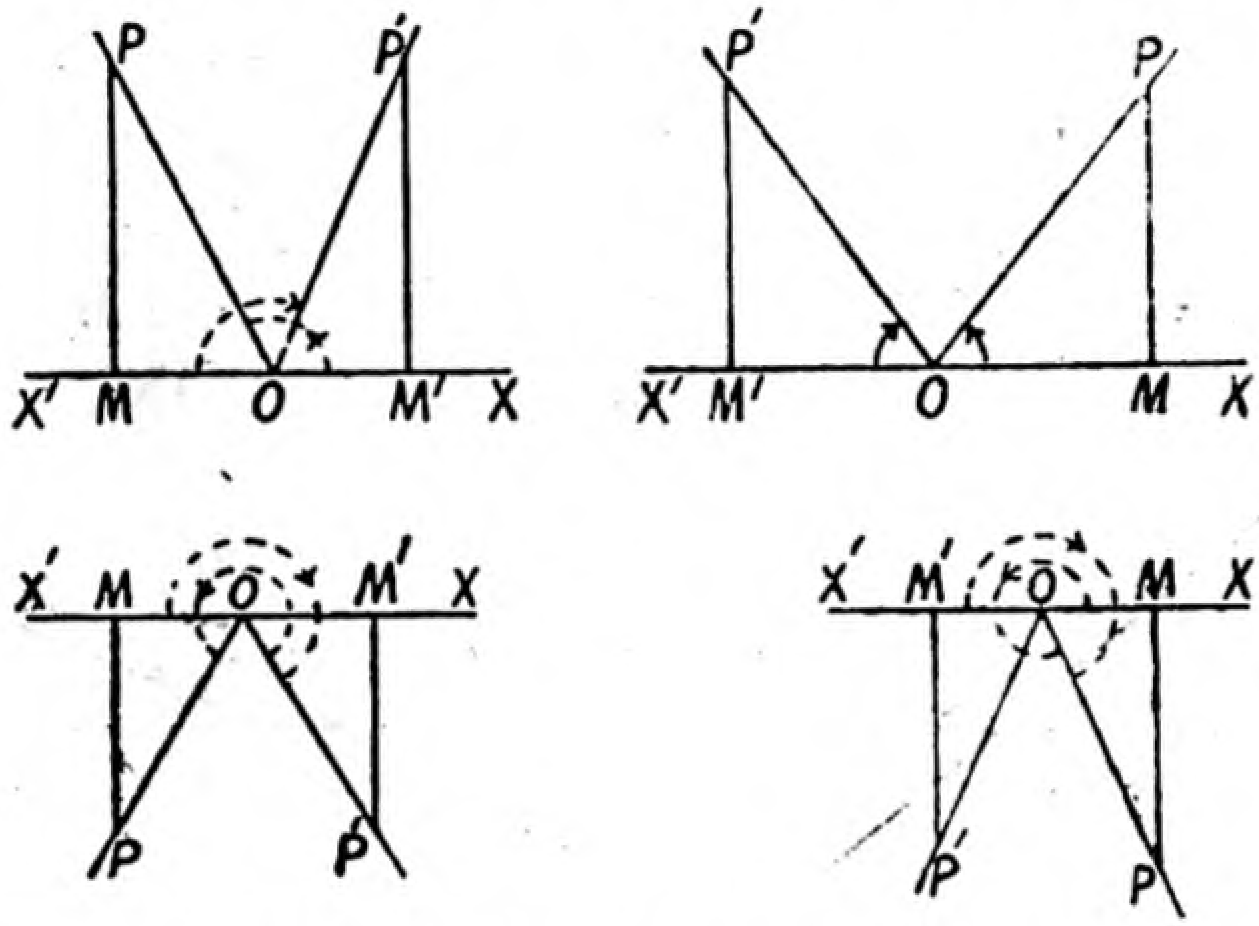
$$\cot(90°+\theta) = \frac{OM'}{M'P'} = -\frac{MP}{OM} = -\tan\theta\,;$$

$$\sec(90°+\theta) = \frac{OP'}{OM'} = -\frac{OP}{MP} = -\operatorname{cosec}\theta\,;$$

$$\text{and } \operatorname{cosec}(90°+\theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

30. ***To compare the trigonometrical ratios of θ and $180°-\theta$; i.e., of supplementary angles.***

Let the revolving line OP starting from OX revolve and describe the angle XOP equal to θ. Then let it coincide with OX′ and then revolve in the opposite direction through θ to the position OP′ so that $\angle XOP'$ is $180°-\theta$.



From any point P in OP, draw MP ⊥ OX or OX′. From OP′ cut off OP′=OP. Draw P′M′ ⊥ OX or OX′. Then triangles OMP and OM′P′ are evidently congruent. Therefore having regard to signs of lines, we have

$OP'=OP$; $OM'=-OM$ and $M'P'=MP$,

$$\therefore \sin(180°-\theta)=\frac{M'P'}{OP'}=\frac{MP}{OP}=\sin\theta;$$

$$\cos(180°-\theta)=\frac{OM'}{OP'}=-\frac{OM}{OP}=-\cos\theta$$

$$\tan(180°-\theta)=\frac{M'P'}{OM'}=-\frac{MP}{OM}=-\tan\theta$$

$$\cot(180°-\theta)=\frac{OM'}{M'P'}=-\frac{OM}{MP}=-\cot\theta$$

$$\sec(180°-\theta)=\frac{OP'}{OM'}=-\frac{OP}{OM}=-\sec\theta$$

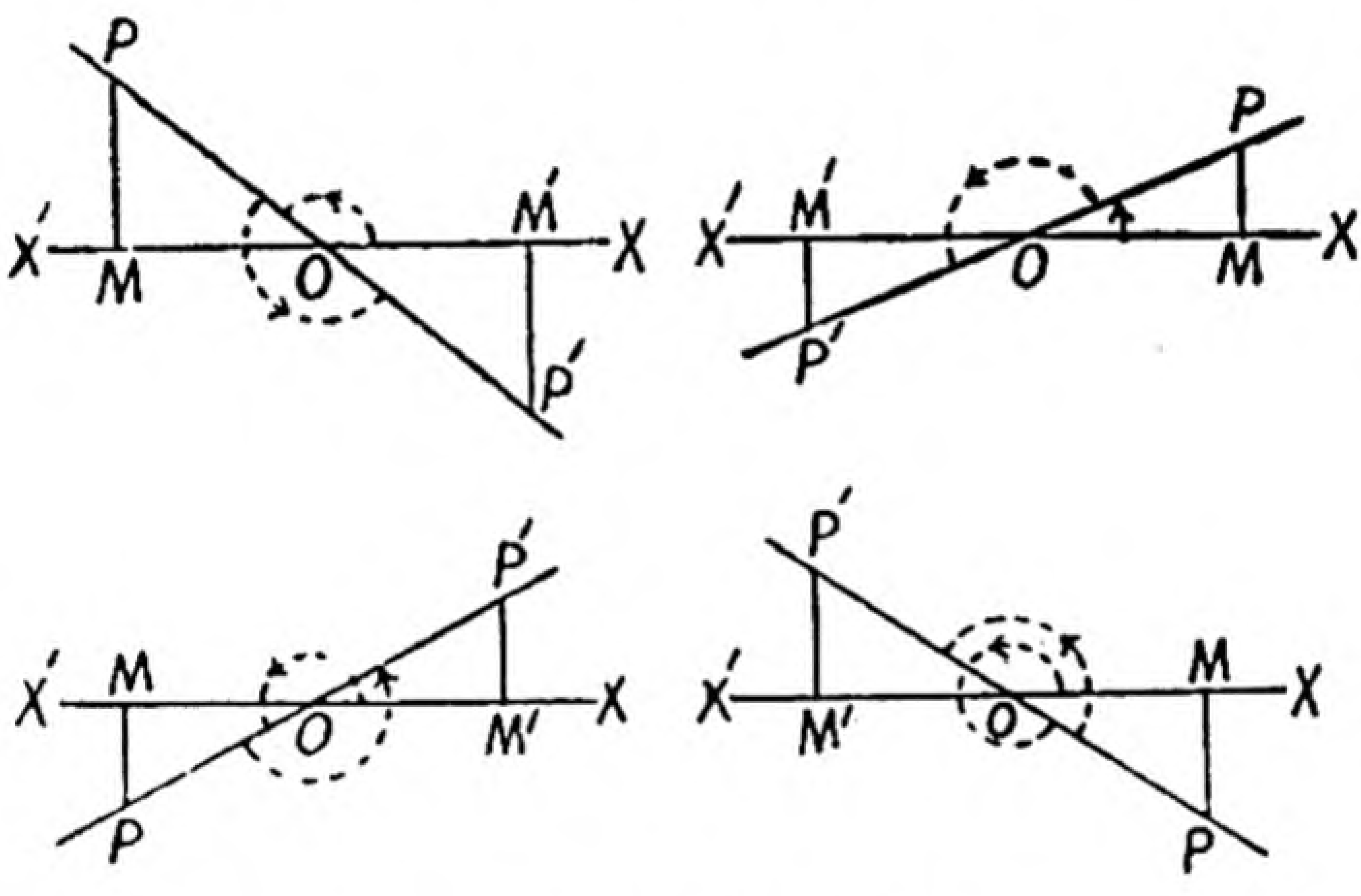
and $$\operatorname{cosec}(180°-\theta)=\frac{OP'}{M'P'}=\frac{OP}{MP}=\operatorname{cosec}\theta.$$

31. *To compare the trigonometrical ratios of* θ *and* $180°+\theta$.

Let the revolving line OP starting from OX revolve and describe an angle XOP equal to θ. Then let it revolve from OP in the positive direction to OP′ through 180° so that $\angle XOP'$ is $180°+\theta$.

From any point P in OP draw $PM \perp OX$ or OX′. Cut off $OP'=OP$ from OP′. Draw $P'M' \perp OX$ or OX′. Then the triangles OMP and OM′P′ are evidently congruent. Therefore having regard to signs of lines, we have

$OP'=OP$; $M'P'=-MP$ and $OM'=-OM$.



$$\therefore \quad \sin(180°+\theta)=\frac{M'P'}{OP'}=-\frac{MP}{OP}=-\sin\theta\,;$$

$$\cos(180°+\theta)=\frac{OM'}{OP'}=-\frac{OM}{OP}=-\cos\theta;$$

$$\tan(180°+\theta)=\frac{M'P'}{OM'}=\frac{-MP}{-OM}=\frac{MP}{OM}=\tan\theta$$

$$\cot(180°+\theta)=\frac{OM'}{M'P'}=\frac{-OM}{-MP}=\frac{OM}{MP}=\cot\theta$$

$$\sec(180°+\theta)=\frac{OP'}{OM'}=-\frac{OP}{OM}=-\sec\theta;$$

$$\text{and } \operatorname{cosec}(180°+\theta)=\frac{OP'}{M'P'}=-\frac{OP}{MP}=-\operatorname{cosec}\theta.$$

Periodicity of tan θ and cot θ. It follows that tan θ assumes the same value when θ is increased by 180° or π. This is expressed by saying that tan θ *is a periodic function of θ, the period being 180° or π.* Similar is the case with cotangent.

32. *To compare the trigonometrical ratios of θ and $\theta \pm 360°$.*

In this case the two positions of the revolving line OP and OP′ coincide in whichever quadrant the angle θ may lie. Hence the quantities MP, OP and OM remain the same for the two angles θ and $\theta \pm 360°$; *i.e.*,

$$\sin(\theta \pm 360°) = \sin\theta$$
$$\cos(\theta \pm 360°) = \cos\theta, \text{ and so on.}$$

Periodicity of Circular Functions. It follows that *the addition or subtraction of any integral multiple of 360° to, or from an angle θ does not alter its trigonometrical ratios.* This is expressed by saying that **the trigonometrical ratios are periodic, the period being 360° or 2π; except for tan θ and cot θ for which the peoriod, as shown already, is 180° or π.**

Ex. 1. Find the value of (*i*) sin 1650° (*ii*) tan 840°.

(*i*) $\sin 1650° = \sin(4 \times 360° + 210°) = \sin 210°$
$= \sin(180° + 30°)$.
$= -\sin 30° = -\frac{1}{2}$.

(*ii*) $\tan 840° = \tan(2 \times 360° + 120°) = \tan 120°$
$= \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$.

Ex. 2. Prove that $\sin 600° \cos 330° + \cos 120° \sin 150° = -1$.

The L.H.S. $= \sin(600° - 360°)\cos(360° - 30°) + \cos(180° - 60°)\sin(180° - 30°)$

$$= \sin 240° \cos 30° - \cos 60° \sin 30°$$
$$= \sin(180° + 60°)\cos 30° - \cos 60° \sin 30°$$
$$= -\sin 60° \cos 30° - \cos 60° \sin 30°$$
$$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = -1.$$

Ex. 3. If A, B, C are the angles of a triangle, prove that $\sin\frac{A}{2} = \cos\frac{B+C}{2}$ and $\sin A = \sin(B+C)$.

As $A+B+C=\pi$, or $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$.

$$\frac{A}{2}=\frac{\pi}{2}-\left(\frac{B}{2}+\frac{C}{2}\right)$$

$$\therefore \quad \sin\frac{A}{2}=\sin\left[\frac{\pi}{2}-\left(\frac{B}{2}+\frac{C}{2}\right)\right]=\cos\left(\frac{B}{2}+\frac{C}{2}\right)$$

Again, $A=\pi-(B+C)$

$\therefore$ $\sin A=\sin[\pi-(B+C)]=\sin(B+C)$

Ex. 4. Express the following angles in the form $k\times 90^\circ\pm\alpha$, where k is an integer or zero and α is a positive angle not greater than 45°; (*i*) 235° (*ii*) 554° (*iii*) -416°.

(*i*) $235^\circ=2\times 90^\circ+56^\circ=3\times 90^\circ-35^\circ$

(*ii*) $554^\circ=6\times 90^\circ+14^\circ$

(*iii*) $-416^\circ=-4\times 90^\circ-56^\circ=-5\times 90^\circ+34^\circ$.

Note. Observe that the method employed above is perfectly general, so that any given angle can be expressed in the form $k\frac{\pi}{2}\pm\alpha$, where α is positive angle not greater than $\frac{\pi}{4}$.

Ex. 5. Show that $\sin(n\pi\pm\alpha)=\pm\sin\alpha$ or $\mp\sin\alpha$ according as n is an even or odd integer.

Let n be an even integer, say $2m$, m being positive or negative.

Then $\sin(n\pi\pm\alpha)=\sin(2m\pi\pm\alpha)=\sin(\pm\alpha)=\pm\sin\alpha$.

Let n be odd, say $2m+1$, where m may be positive or negative.

Then $\sin(n\pi\pm a)=\sin(2m\pi-\pi\pm\alpha)$

$$=\sin(\pi\pm\alpha)=\mp\sin\alpha.$$

33. *The trigonometrical ratios of any angle can be expressed in terms of the trigonometrical ratios of an acute angle not greater than 45°.*

Any angle θ is of the form $k\frac{\pi}{2}\pm\alpha$, where k is an integer (positive or negative) and α is a positive angle not greater than $\frac{\pi}{4}$.

Now k is (*a*) either an even integer (*b*) or an odd integer

(*a*) Let k be an even integer, positive or negative, say $2n$. Then

$$\sin\theta=\sin\left(k\frac{\pi}{2}\pm\alpha\right)=\sin(n\pi\pm\alpha).$$

Now if n is even say, $2p$, then.

$\sin(n\pi\pm\alpha)=\sin(2p\pi\pm\alpha)=\pm\sin\alpha.$

But if n is odd, say $2q+1$, then

$\sin(n\pi\pm\alpha)=\sin(2q\pi+\pi\pm\alpha)=\sin(\pi\pm\alpha)=\mp\sin\alpha.$

(*b*) Next, let k be an odd integer, positive or negative say $2m+1$. Then

$$\sin\theta=\sin\left(k\frac{\pi}{2}\pm\alpha\right)=\sin\left(m\pi+\frac{\pi}{2}\pm\alpha\right)$$
$$=\sin\left(\frac{\pi}{2}+m\pi\pm\alpha\right)=\cos(m\pi\pm\alpha).$$

Now if m is even, say $2p$, then

$\cos(m\pi\pm\alpha)=\cos(2p\pi\pm\alpha)=\cos(\pm\alpha)=\cos\alpha.$

But if m is odd, say $2q+1$, then

$\cos(m\pi\pm\alpha)=\cos(2q\pi+\pi\pm\alpha)=\cos(\pi\pm\alpha)=-\cos\alpha.$

Similarly other trigonometrical ratios can be expressed in terms of those of α.

The positive and negative integers cannot only be divided into groups of even and odd numbers but also in the following manner :—

1. (*a*) Those divisible by 3, *i.e.*, of the type $3m$; (*b*) those which give unity as remainder when divided by 3, *i.e.*, of the type $3m+1$, (*c*) those which give 2 as remainder when divided by 3, *i.e.*, of the type $3m+2$.

2. Similarly $4m$, $4m+1$, $4m+2$, $4m+3$ cover all positive or negative integers, when m is any integer (positive or negative) and so on.

The article 33 above can also be proved as follows :—

$$\sin\theta=\sin\left(k\frac{\pi}{2}\pm\alpha\right),$$

where k is any positive or negative integer and so of the type $4m$, $4m+1$, $4m+2$, $4m+3$.

(*i*) If $k=4m$ then

$$\sin\theta=\sin\left(4m\times\frac{\pi}{2}\pm\alpha\right)$$
$$=\sin(2m\pi\pm\alpha)$$
$$=\sin(\pm\alpha)=\pm\sin\alpha.$$

(*ii*) If $k=4m+1$ then

$$\sin\theta=\sin\left(4m\times\frac{\pi}{2}+\frac{\pi}{2}+\alpha\right)$$

$$=\sin\left(\frac{\pi}{2}\pm\alpha\right)=\cos\alpha.$$

(*iii*) $k=4m+2$ then

$$\sin\theta=\sin[2m\pi+\pi\pm\alpha]$$

$$=\sin(\pi\pm\alpha)=\pm\sin\alpha.$$

(*iv*) If $k=4m+3$ then

$$\sin\theta=\sin\left[2m\pi+\frac{3\pi}{2}\pm\alpha\right]$$

$$=\sin\left(\frac{3\pi}{2}\pm\alpha\right)=-\cos\alpha.$$

Thus the result is proved in case of sine. Exactly in the similar way the same result can be proved for other trigonometrical ratios also.

EXERCISE IX

1. Find the values of :—
   (*i*) $\sin 945°$. (*ii*) $\sin 495°$. (*iii*) $\tan 300°$.
2. Find the values of :—
   (*i*) $\sin(-390)°$. (*ii*) $\cos(-945)°$. (*iii*) $\tan(1140)°$.
3. Find the values of :—
   (*i*) $\operatorname{cosec} 2040°$. (*ii*) $\sec 3060°$. (*iii*) $\cot 720°$.

Express in their simplest form :—

4. $\sin(180°+A)\cos(90°-A)$.
5. $\tan(180°-A)\sec(180°+A)\sin(90°+A)$.

Prove that

6. (*i*) $\sin^2 36°-\sin^2 18°=\sin^2 72°-\sin^2 54°$.
   (*ii*) $\sin 420°\cos 390°+\cos(-660°)\sin(-330°)=1$.
   (*iii*) $\sin 600°\cos 330°+\cos 120°\sin 150°=-1$.
7. $\tan 225°\cot 405°+\tan 405°\cot 675°=0$.

Simplify :

8. $$\frac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)}$$
9. $$\frac{\sin(180°+\theta)\cos(270°-\theta)}{\sin(180°-\theta)\cos(270°+\theta)}$$
10. If $A+B+C=180°$, show that
    $$\tan(A+B)+\tan C=0.$$

11. In a triangle ABC, prove that

$$\cos\frac{A}{2}=\sin\frac{B+C}{2} \text{ and } \sin\frac{A}{2}=\cos\frac{B+C}{2}.$$

12. In a triangle ABC, prove that

$$\sin A=\sin(B+C) \text{ and } \cos A=-\cos(B+C).$$

13. If A, B, C, D are the angles of a quadrilateral, prove that (*i*) $\sin(A+B)+\sin(C+D)=0$.

(*ii*) $\cos(A+B)=\cos(C+D)$.

14. A quadrilateral ABCD is inscribed in a circle. Show that $\sin A=\sin C$ and $\cos B+\cos D=0$.

15. If A, B, C, D be the angles of a cyclic quadrilateral, prove that $\cos A+\cos B+\cos C+\cos D=0$.

16. Prove geometrically that

$$\cos(270^\circ-A)=-\sin A \text{ and } \sin(270^\circ+A)=-\cos A.$$

17. Show that in general

$$\cos(m\pi+\theta)=(-1)^m\cos\theta.$$

18. Prove that $\sin\left(\frac{\pi}{4}+\theta\right)=\cos\left(\frac{\pi}{4}-\theta\right)$.

19. If $\tan(n\pi+\theta)=a$ and $\cos(2m\pi\pm\theta)=b$, show that $b^2(a^2+1)=1$ provided that m and n are integers.

CHAPTER V

VARIATIONS OF TRIGONOMETRICAL RATIOS AND THEIR GRAPHS

34. *To trace the variations of* $\sin\theta$ *as* θ *increases continuously from* 0° *to* 360°, *and to exhibit them graphically.*

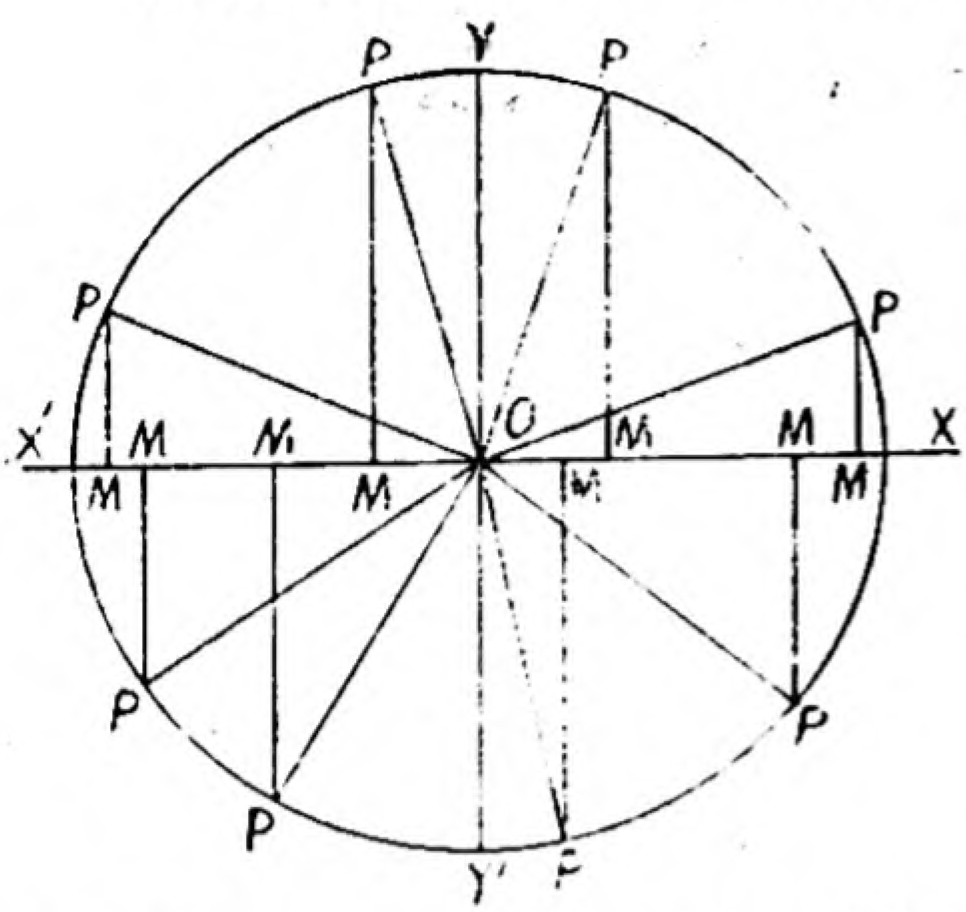
In the figure $\angle XOP=\theta$.

Let the revolving line OP be of constant length, say 1.

Now $\sin\theta=\frac{MP}{OP}$.

OP being constant, we have to observe the variations of MP.

First Quadrant. In the first quadrant when $\theta=0°$. M and P coincide and therefore MP is zero, so that $\sin 0°=0$. As θ increases, MP and therefore $\sin\theta$ increases, till when $\theta=90°$. MP=OP and hence $\sin 90°=1$. Thus in the first quadrant as θ varies from $0°$ to $90°$, $\sin\theta$ is positive and varies from 0 to 1, *i.e.*, increases from 0 to 1.



Second Quadrant. As θ increases, MP is positive and decreases so that $\sin\theta$ is positive and decreases; and when $\theta=180°$, MP vanishes and therefore $\sin 180°=0$.

Thus in the second quadrant $\sin\theta$ varies from 1 to 0 *i.e.*, decreases from 1 to 0 and is positive because MP is positive.

Third Quadrant. As θ increases, MP is negative and increases in magnitude so that $\sin\theta$ is negative and increases in magnitude.

When $\theta=270°$, MP=OP in magnitude and $\therefore$ $\sin 270° = -1$.

Thus in the third quadrant $\sin\theta$ varies from 0 to -1 and is negative because MP is negative.

Fourth Quadrant. As θ increases, MP is negative and decreases in magnitude, so that $\sin\theta$ is negative and decreases in magnitude. When $\theta=360°$, MP is zero, so that $\sin 360°=0$.

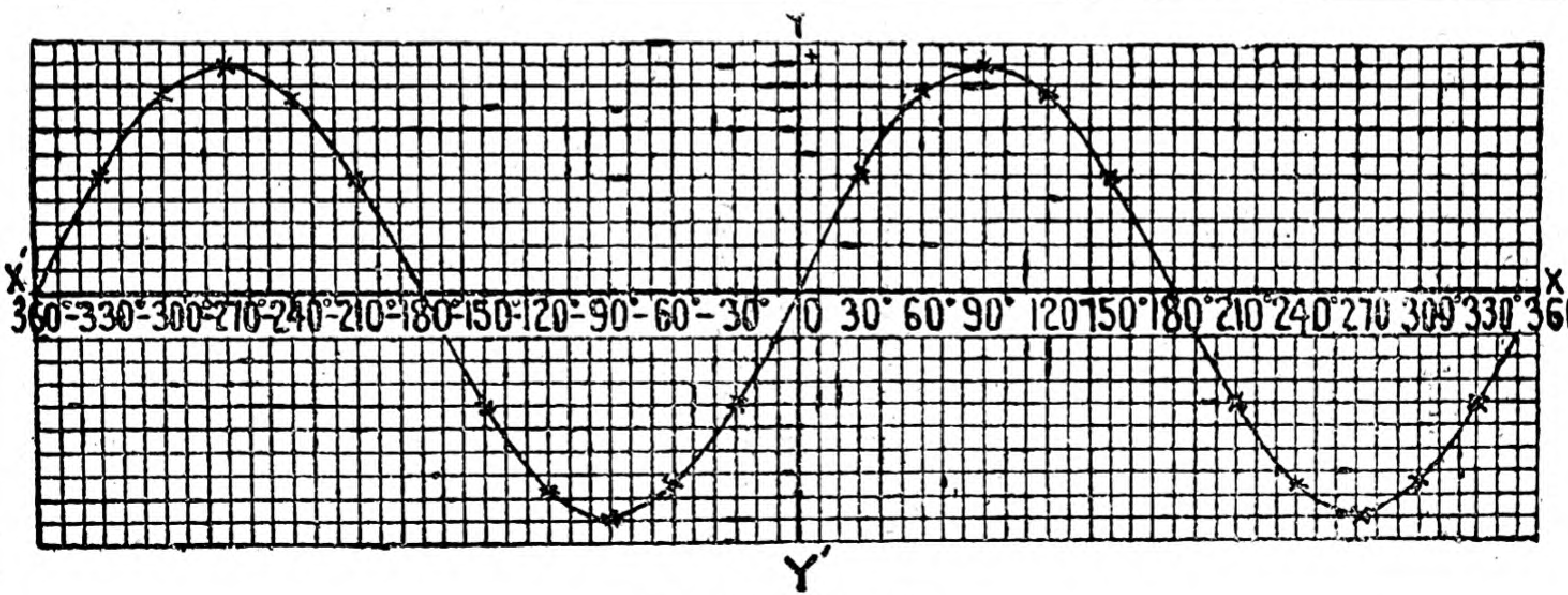
Thus in the fourth quadrant $\sin\theta$ varies from -1 to 0, and is negative, because MP is negative.

Note 1.—It follows that $\sin\theta$ is never greater than unity and that it is capable of assuming any value between 1 and -1.

Note 2.—It also follows that there are two angles lying between $0°$ and $360°$, which have a given sine; if the given sine is positive, the two angles lie between $0°$ and $180°$ and if the given sine is negative, the angles lie between $180°$ and $360°$.

TABLE FOR THE SINE GRAPH

$x=$	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$\sin x=$	0	·5	·87	**1**	·87	5	0	−·5	−·87	−1	−·87	−·5	0
$x=$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\sin x=$	·0	·5	·87	1	·87	·5	·0	−·5	−·87	−1	−·87	−·5	0



The Sine Graph

35. *To trace the variations of cos θ as θ varies continuously from 0° to 360° and to exhibit them graphically.*

Referring to the figure of Article 34, $\cos\theta=\dfrac{OM}{OP}$.

So the variations in cos θ depend upon the variations in the values of OM.

First Quadrant. In the first quadrant when $\theta=0°$, M and P coincide and therefore OM = OP and hence $\cos 0°=1$. As θ increases, OM and therefore cos θ decreases, till when $\theta=90°$, OM is zero, and hence $\cos 90°=0$.

Thus in the first quadrant cos θ varies from 1 to 0, *i.e.*, **decreases and is positive, because OM is positive.**

Second Quadrant. As θ increases, OM is negative and increases in magnitude, consequently cos θ is negative and increases in magnitude, till when $\theta=180°$, OM=OP in magnitude and hence $\cos 180°=-1$.

Thus in the second quadrant cos θ varies from 0 to -1 and is negative because OM is negative.

Third Quadrant. As θ increases, OM is still negative and decreases in magnitude; so that cos θ is negative and decreases in magnitude, till when $\theta=270°$, OM is zero and therefore $\cos 270°=0$.

Thus in the third quadrant cos θ varies from -1 to 0 and is negative, because OM is negative.

Fourth Quadrant. As θ increases, OM is positive and increases so that cos θ is positive and increases, till when $\theta=360°$, OM=OP, and therefore $\cos 360°=1$.

Thus in the fourth quadrant cos θ varies from 0 to 1 and is positive, because OM is positive.

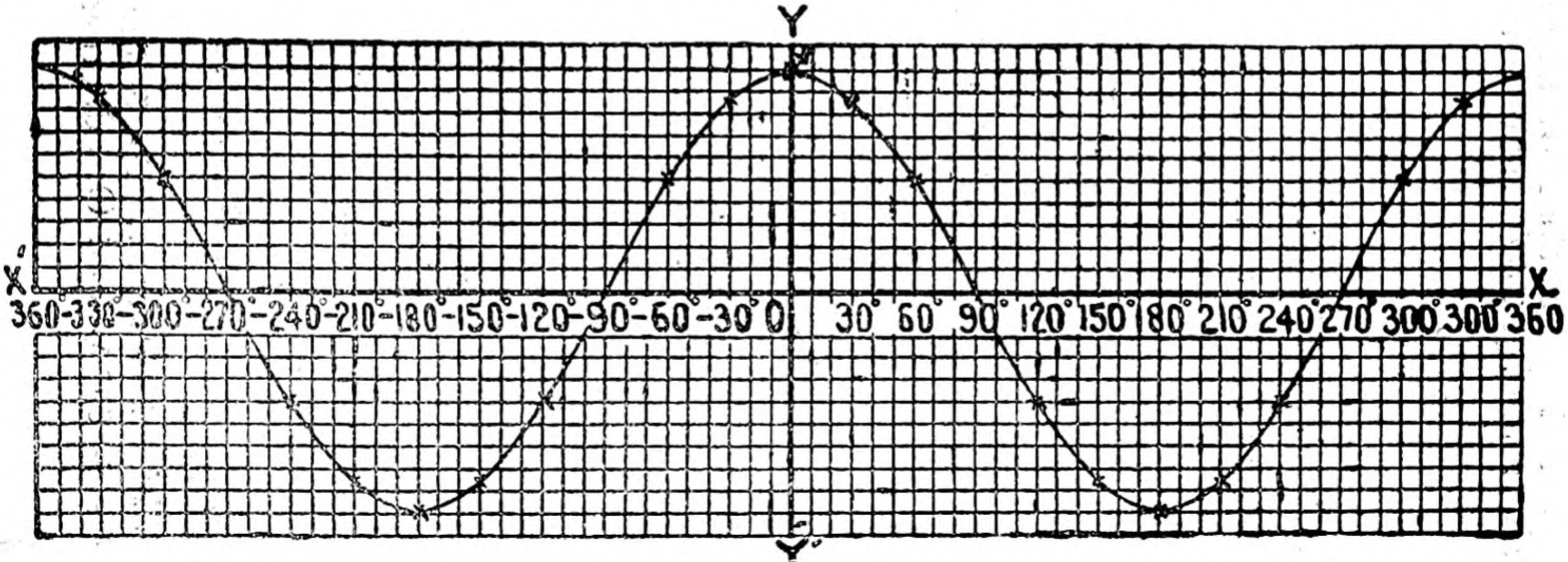
Note 1.—It follows that cos θ is never greater than unity and that it is capable of assuming any value lying between and 1 and -1.

Note 2.—It also follows that there are two angles lying between 0° and 360°, which have a given cosine; if the given cosine is prositive, one of the angles lies between 0° and 90° and the other between 270° and 360° but if the given cosine is negative; then the two angles lie between 90° and 270°.

Note 3.—It may be observed that sin θ is less than cos θ for any value of θ between 0° and 45° and greater for any value between 45° and 90°.

TABLE FOR THE COSINE GRAPH

$x=$	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$\cos x=$	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1
$x=$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\cos x=$	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1



The Cosine Graph

36. *To trace the variations of tan θ as θ varies conitnuously from 0° to 360° and to exhibit them graphically.*

Referring to the figure of article 34, $\tan\ \theta = \frac{MP}{OM}$.

So the variations in tan θ depend upon the variations in both MP and OM.

First Quadrant. In the first quadrant when $\theta = 0°$, M and P coincide so that MP is zero and OM=OP and therefore $\tan 0° = 0$.

As θ increases, MP increases, and OM decreases and therefore on both these accounts tan θ increases. When OP has turned through an angle which is slightly less than a right angle so that P is very near to Y, OM is very small and MP is very nearly equal to OP or 1 and consequently tan θ is very large ; therefore by taking an angle sufficiently near to 90°, we can make the tangent as large as we please. This fact is, for the sake of brevity expressed thus : the tangent of 90° is infinite.

In the first quadrant, therefore, tan θ increases from 0 to ∞ (infinity), and is positive, because MP and OM are both positive.

Second Quadrant. As θ increases slightly, OM becomes negative while remaining small, and MP is positive and very nearly equal to OP or 1, so that the corresponding tangent is very large and negative. As θ increases in magnitude, OM increases in magnitude while MP decreases, so that tan θ decreases in magnitude, till when $\theta = 180°$, MP is zero, and OM=OP=1 and therefore $\tan 180° = 0$

In the second quadrant, therefore, tan θ varies from $-\infty$ to 0 and is negative, because OM is negative and MP is positive.

Third Quadrant. As θ increases, OM and MP both become negative and OM decrease in magnitudes while MP

increases in magnitude, so that tan θ is positive and increases, till when $\theta \to 270°$, OM$\to 0$ and MP$\to$OP$=1$ and $\therefore$ tan 270° is infinite.

In the third quadrant, therefore, tan θ varies from 0 to ∞ and is positive, because OM and MP are both negative.

Fourth Quadrant. As θ increases slightly, OM is small but becomes positive, while MP remains negative, and very nearly equal to OP or 1 so that the corresponding tangent is very large and negative. As θ increases, OM increases and MP decreases in magnitude, so that tan θ decreases in magnitude, till when $\theta=360°$, MP is zero and OM$=$OP$=1$ and therefore tan $360°=0$.

In the fourth quadrant, therefore, tan θ varies from $-\infty$ to 0 and is negative, because OM is positive and MP is negative.

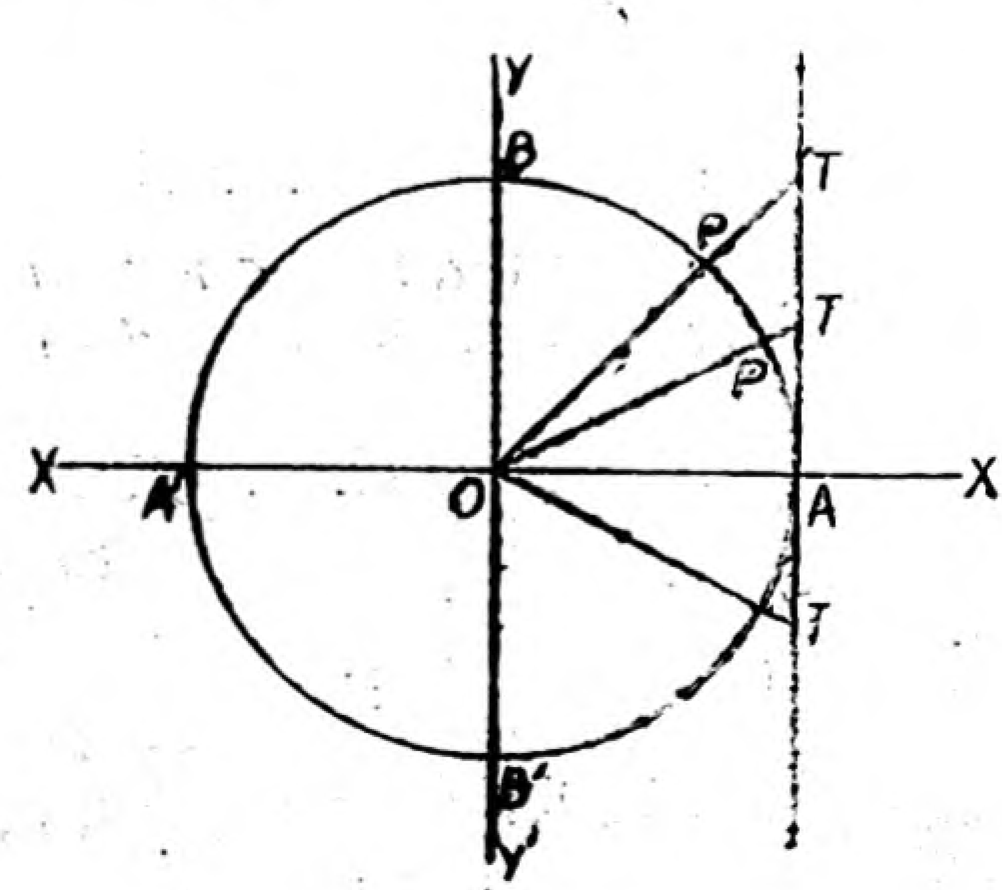
Note 1.—It follows that tan θ is capable of assuming any real value whatever.

Note 2.—It also follows thot there are two angles lying between 0° and 360°, which have a given tangent ; if the given tangent is positive, one of the angles lies between 0° and 90° and the other between 180° and 270°, but if the given tangent is negative, then one of the angles lies between 90° and 180° and the other between 270° and 360°.

Another Method.—In the Fig. of Art. 34, let OP meet the tangent at A in T.

Then tan $\theta = \dfrac{AT}{OA} = AT$,

$\because \quad OA=1$.



Hence AT represents the tangent of the angle XOP.

tan θ is positive when T is above A and is negative when T is below A.

As the angle θ increases from 0 to $\frac{\pi}{2}$, AT is positive and increases from 0 to ∞ because

when $\theta=\frac{\pi}{2}$, OT coincides with OY which is ∥ to the tangent at A.

$\therefore$ tan θ is positive and increases from 0 to ∞.

When the angle θ is a little less than $\frac{\pi}{2}$, T falls above A and $\therefore$ tan θ is positive and very large; when θ is a little $>\frac{\pi}{2}$, T falls below A and tan θ is negative and very large; $\therefore$ when θ passes through the value $\frac{\pi}{2}$, tan θ suddenly changes from $+\infty$ to $-\infty$.

From $\frac{\pi}{2}$ to π. AT is negative and increases from $-\infty$ to 0; $\therefore$ tan θ increases from $-\infty$ to 0.

From π to $\frac{3\pi}{2}$, AT is positive and increases from 0 to ∞. $\therefore$ tan θ is positive and increases from 0 to ∞.

When θ passes through the value $\frac{3\pi}{2}$, tan θ again suddenly changes from $+\infty$ to $-\infty$.

From $\frac{3\pi}{2}$ to 2π, AT is negative and increases from $-\infty$ to 0.

$\therefore$ tan θ increases from $-\infty$ to 0.

37. *To trace the variations of cot θ as θ varies continuously from 0° to 360° and to exhibit them graphically.*

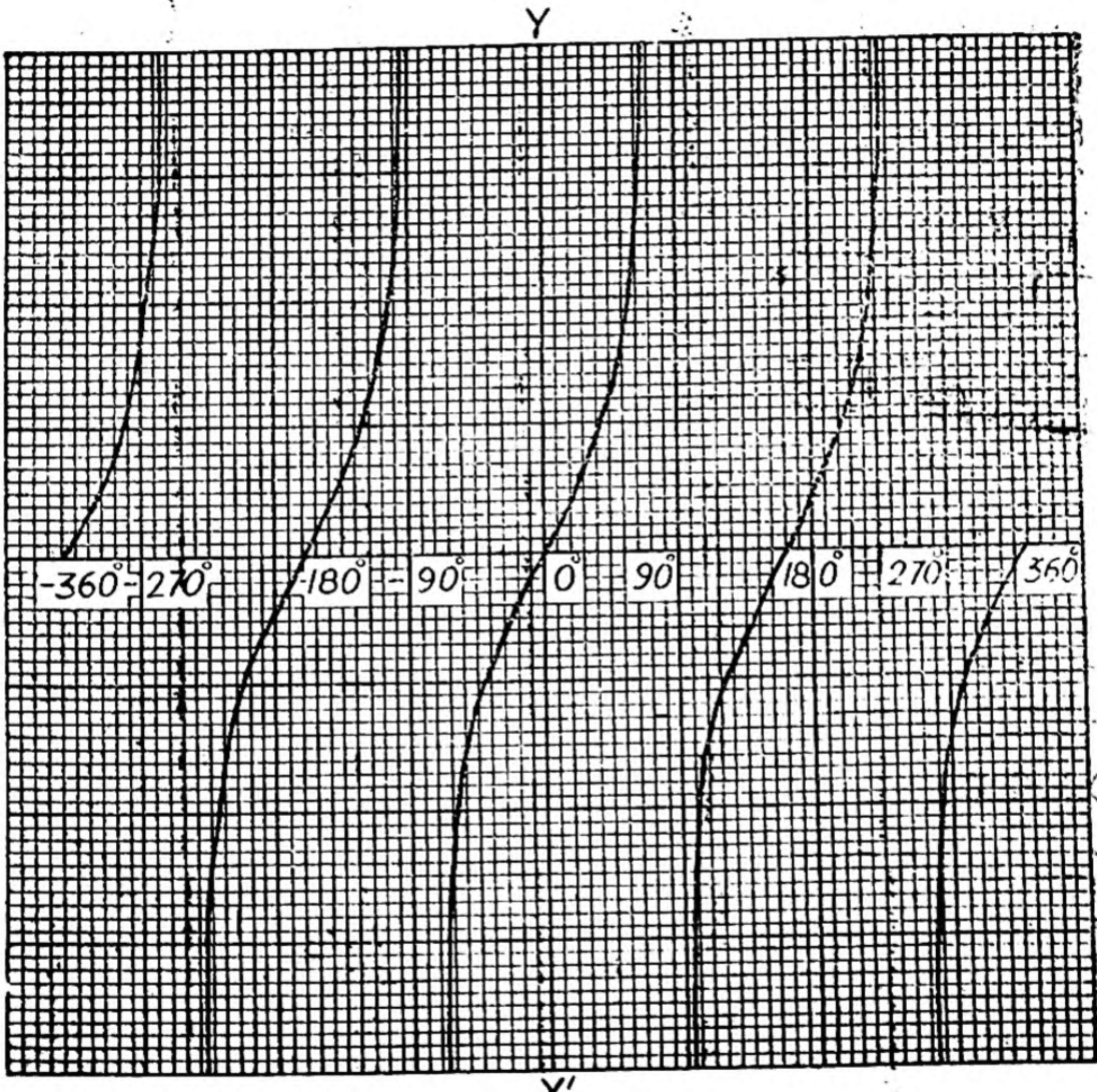
Referring to the figure of Article 34, cot $\theta=\frac{OM}{MP}$.

So the variations in cot θ depend upon the variations in both OM and MP.

First Quadrant. When θ is very small, MP is positive and very small and OM is very nearly equal to OP or 1. As $\theta\to 0$, MP$\to$0 and OM$\to$OP or 1 so that cot 0° is infinite.

TABLE FOR THE TANGENT GRAPH

$x =$	$-360°$	$-330°$	$-300°$	$-270°-0°$	$-270°+0°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°-0°$	$-90°+0°$	$-60°$	$-30°$	$0°$
$\tan x =$	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0
$x =$	0°	30°	60°	$90°-0°$	$90°+0°$	120°	150°	180°	210°	240°	$270°-0°$	$270°+0°$	300°	330°	360°
$\tan x =$	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0



The Tangent Graph

As θ increases, OM decreases and MP increases; so that cot θ decreases, till when $\theta=90°$, OM is zero and MP=OP =1 and consequently cot $90°=0$.

Thus in the first quadrant cot θ varies from ∞ to 0, and is positive, because OM and MP are both positive.

Second Quadrant. As θ increases, OM becomes negative and increases in magnitude, while MP is positive and decreases; so that cot θ is negative and increases in magnitude, till when θ is very near to 180°, MP is very small and OM is very nearly equal to OP or 1 and, therefore, cot 180° is negative and infinite.

Thus in the second quadrant cot θ varies from 0 to $-\infty$ and is negative because OM is negative and MP is positive.

Third Quadrant. As θ is slightly greater than 180°, OM and MP both become negative and MP is small, and OM is very nearly equal to OP or 1, so that cot θ is positive and infinite. As θ increases, MP increases in magnitude while OM decreases in magnitude so that cot θ is positive and decreases in magnitude, till when $\theta=270°$, OM is zero and MP=OP or 1 and therefore cot $270°=0$.

Thus in the third quadrant cot θ varies from $+\infty$ to 0 and is psoitive, because OM and MP are both negative.

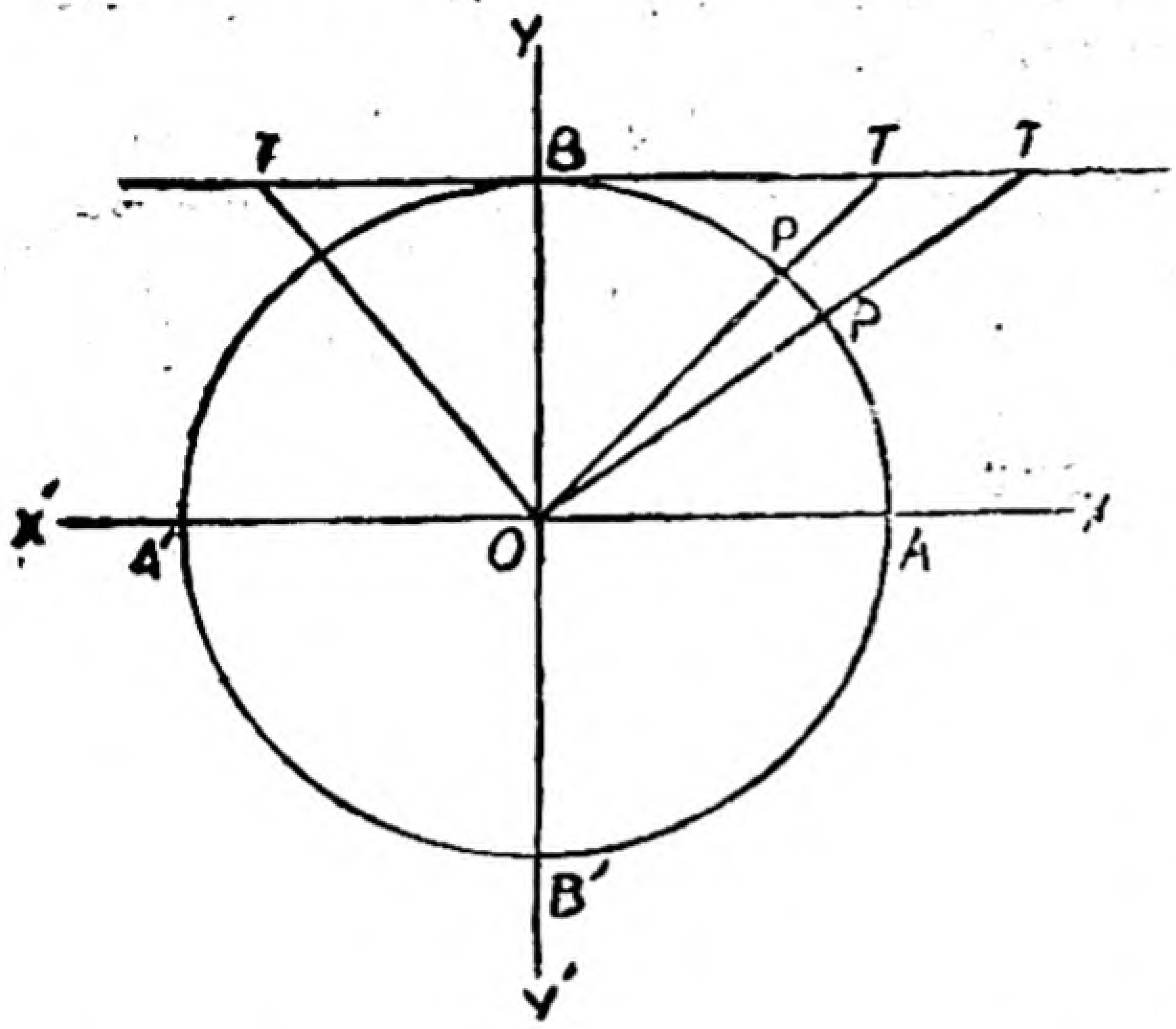
Fourth Quadrant. As θ increases, OM becomes positive and increases while MP is negative and decreases in magnitude, so that θ is negative, and increases in magnitude, till when θ is very near to 360°, MP is small and OM is very nearly equal to OP or 1 and therefore cot 360° is negative and infinite.

Thus in the fourth quadrant cot θ varies from 0 to $-\infty$ and is negative, because OM and MP have opposite sign.

Note 1.—It follows that cot θ is capable of assuming any real value whatever.

Note 2.—It also follows that there are two angles lying between 0° and 360°, which have a given cotangent; if the given cotangent is positive, one of the angles lies between 0° and 90°, and the other between 180° and 270°; but if the given cotangent is negative then one of the angles lies between 90° and 180°, and the other between 270° and 360°.

Another Method. In the fig. of Art. 34, let OP meet the tangent at B in T.



$\cot\theta = \cot XOP = \cot OTB$. [$\because$ BT is $\parallel$ to OX.]

$= \dfrac{BT}{OB} = BT$, as $OB = 1$.

$\therefore \cot\theta = BT$.

Hence BT represents cotangent of the angle θ.

$\cot\theta$ is positive when T is to the right of B or O, and it is negative when T is to the left of B.

As θ increases from 0 to $\frac{\pi}{2}$, BT is positive and decreases from ∞ to 0, $\therefore$ $\cot\theta$ is positive and decreases from ∞ to 0.

From $\frac{\pi}{2}$ to π, BT is negative and decreases from 0 to $-\infty$, $\therefore$ $\cot\theta$ is negative and decreases from 0 to $-\infty$.

As θ passes through the value π, $\cot\theta$ suddenly changes from $-\infty$ to $+\infty$.

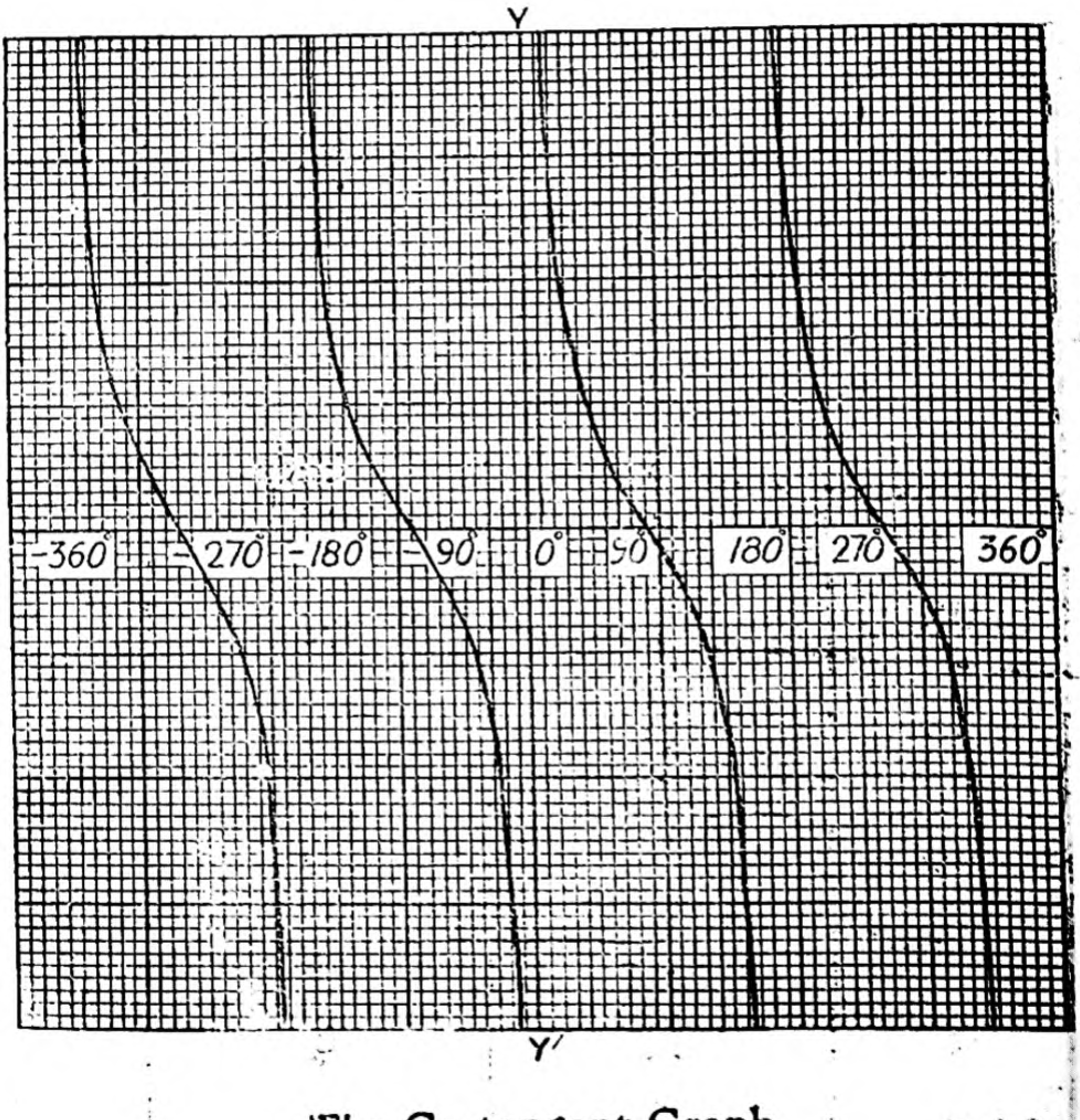
From π to $\frac{3\pi}{2}$, BT is positive and decreases from ∞ to 0, $\therefore$ $\cot\theta$ is positive and decreases from ∞ to 0.

From $\frac{3\pi}{2}$ to 2π, BT is negative and decreases from 0 $-\infty$, $\therefore$ $\cot\theta$ is negative and decreases from 0 to $-\infty$.

As θ passes through the value 2π, $\cot\theta$ suddenly changes from $-\infty$ to $+\infty$.

TABLE FOR THE COTANGENT GRAPH

$x=$	$-360°+0°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°-0°$	$-180°+0°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$\cot x=$	$+\infty$	1·7	·58	0	−·58	−·17	$-\infty$	$+\infty$	1·7	·58	0	−·58	−1·7	$-\infty$
$x=$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$ $-0°$	$180°$ $+0°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$ $-0°$
$\cot x=$	$+\infty$	1·7	·58	0	−·58	−1·7	$-\infty$	$+\infty$	1·7	·58	0	−·58	−1·7	$-\infty$



The Co-tangent Graph

38. *To trace the variations of secant θ as θ varies continuously from 0° to 360°, and to exhibit them graphically.*

Referring to the figure of Article 34, sec $\theta = \dfrac{OP}{OM}$.

OP being constant, we have to observe the variations of OM.

First Quadrant. When θ is zero, M and P coincide, so that OM=OP and consequently sec $0° = 1$. As θ increases, OM decreases so that sec θ increases; when θ is very near to 90°, OM is very near to 0 and therefore, sec 90° is infinite.

Thus in the first quadrant sec θ varies from 1 to ∞ and is positive because OM is positive.

Second Quadrant. As θ increases slightly, OM becomes negative and remains small, so that sec θ is negative and infinite. As θ increases, OM increases in magnitude so that sec θ is negative and decreases in magnitude till when $\theta = 180°$, OM equals OP in magnitude and therefore sec $180° = -1$.

Thus in the second quadrant sec θ varies from $-\infty$ to -1 is negative, because OM is negative.

Third Quadrant. As θ increases, OM remains negative and decreases in magnitude; so that sec θ is negative and increases in magnitude; when θ comes nearer and nearer to 270°, OM becomes smaller and smaller: therefore sec θ becomes larger and larger; hence sec 270° is infinite and negative.

Thus in the third quadrant sec θ varies from -1 to $-\infty$ and is negative, because OM is negative.

Fourth Quadrant. As θ increases slightly, OM becomes positive and remains small and therefore sec θ is positive and infinite. As θ increases, OM increases and therefore sec θ decreases till when $\theta = 360°$, OM=OP and therefore sec $360° = 1$.

Thus in the fourth quadrant sec θ varies from ∞ to 1 and is positive, because OM is positive.

Note 1.—It follows that sec θ never lies between 1 and -1 and that it is capable of assuming any real value not lying between 1 and -1.

Note 2.—It also follows that there are two angles lying between 0° and 360°, which have a given secant; if the given secant is positive, one of the angles lies between 0° and 90° and the other between 270° and 360° but if the given secant is negative, then the angles lie between 90° and 270°.

Another Method. Take the figure of Art. 36.

In this case $\sec\theta = \dfrac{OT}{OA} = T, \because OA = 1$;

$\therefore$ TO represents the secant of the angle XOP.

Sec θ is negative if OP meets the tangent at A, when OP (*i.e.*, OP) is produced backwards.

As the angle θ increases from 0 to $\dfrac{\pi}{2}$, OT is positive and increases from 1 to ∞, $\therefore$ sec θ is positive and increases from 1 to ∞.

When θ passes through the value $\dfrac{\pi}{2}$, sec θ suddenly changes from $+\infty$ to $-\infty$.

From $\dfrac{\pi}{2}$ to π, OT is negative and increases from $-\infty$ to -1. $\therefore$ sec θ is negative and increases from $-\infty$ to -1.

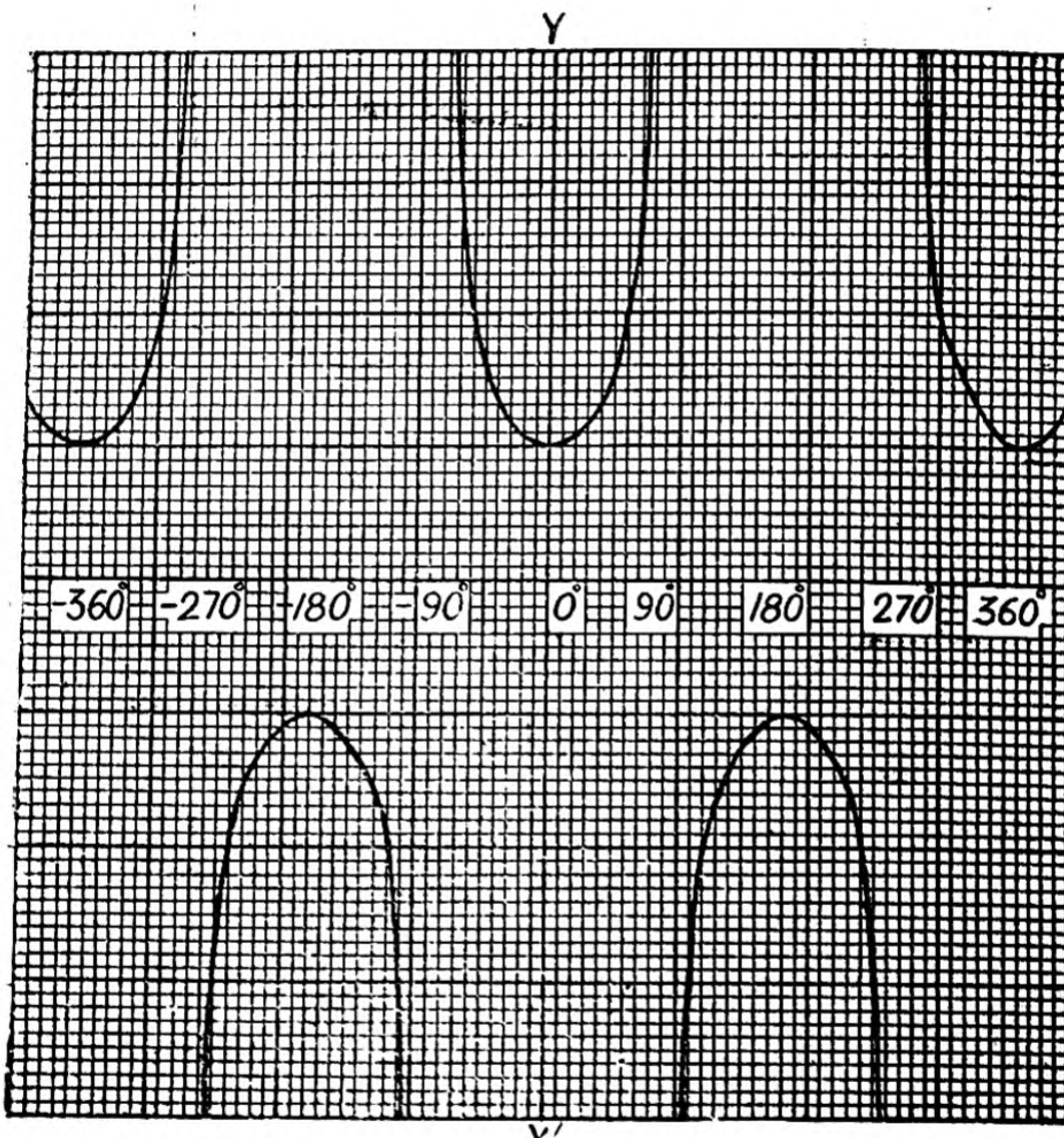
From π to $\dfrac{3\pi}{2}$ OT is negative and decreases from -1 to $-\infty$, $\therefore$ sec θ is negative and decreases from -1 to $-\infty$.

As θ passes through the value $\dfrac{3\pi}{2}$, sec θ suddenly changes from $-\infty$ to $+\infty$.

From $\dfrac{3\pi}{2}$ to 2π, OT is positive and decreases from $+\infty$ to $+1$, $\therefore$ sec θ is positive and decreases from $+\infty$ to $+1$.

TABLE FOR THE SECANT GRAPH

$x =$	$\sec x =$	$x =$	$\sec x =$
$-360°$	1	$0°$	1
$-330°$	1·2	$30°$	1·2
$-300°$	2	$60°$	2
$-270°-0°$	$+\infty$	$90°-0°$	$+\infty$
$-270°+0°$	$-\infty$	$90°+0°$	$-\infty$
$-240°$	-2	$120°$	-2
$-210°$	$-1·2$	$150°$	$-1·2$
$-180°$	-1	$180°$	-1
$-150°$	$-1·2$	$210°$	$-1·2$
$-120°$	-2	$240°$	-2
$-90°-0°$	$-\infty$	$270°-0°$	$-\infty$
$-90°+0°$	$+\infty$	$270°+0°$	$+\infty$
$-60°$	2	$300°$	2
$-30°$	1·2	$330°$	1·2
$0°$	1	$360°$	1



The Secant Graph

39. *To trace the variations of cosec θ as θ varies continuously from 0 to 360° and to exhibit them graphically.*

Referring to the figure of Art. 34,

$$\text{cosec } \theta = \frac{OP}{MP}.$$

OP being constant, we have to observe the variations of MP.

First Quadrant. When θ is very small, MP is positive and very small and as $\theta \to 0$, $MP \to 0$ and $\therefore$ cosec $\theta \to \infty$, so that cosec θ is infinite to start with. As θ increases, MP increases and therefore cosec θ decreases, till when $\theta = 90°$. MP equals OP and therefore cosec $90° = 1$.

Thus in the first quadrant cosec θ varies from ∞ to 1 and is positive because MP is positive.

Second Quadrant. As θ increases, MP is positive and decreases, so that the cosec θ increases; when θ approaches nearer and nearer to 180°, MP approaches zero, so that cosec 180° is infinite.

Thus in the second quadrant cosec θ varies from 1 to ∞ and is positive because MP is positive.

Third Quadrant. As θ increases slightly, MP is small but becomes negative, so that cosec θ is negative and infinite.

As θ increases, MP increases in magnitude so that cosec θ decreases in magnitude till when $\theta = 270°$, MP equals OP in magnitude and therefore cosec $270° = -1$.

Thus in the third quadrant cosec θ varies from $-\infty$ to -1 and is negative, because MP is negative.

Fourth Quadrant. As θ increases, MP remains negative and decreases in magnitude; so that cosec θ is negative and increases in magnitude. When θ approaches nearer and nearer to 360°, MP approaches zero and therefore cosec θ becomes larger and larger; hence cosec 360° is negative and infinite.

Thus in the fourth quadrant cosec θ varies from -1 to $-\infty$ and is negative, because MP is negative.

Note 1—It follows that cosec θ never lies between 1 and -1 and that it is capable of assuming any real value not lying between 1 and -1.

Note 2.—It also follows that there are two angles lying between 0° and 360° which have a given cosecant; if the given cosecant is positive, the angles lie between 0° and

$180°$; but if the given cosecant is negative, the angles lie between $180°$ and $360°$.

Another Method. Take the figure of Art. 37.

In this case cosec θ=OT.

$\therefore$ OT represents the cosecant of θ; cosec θ is negative when the bounding line OT or the positive angle meets the tangent at B at a point in OT produced backwards.

From 0 to $\frac{\pi}{2}$, OT is positive and decreases from ∞ to 1.

$\therefore$ cosec θ decreases from ∞ to 1.

From $\frac{\pi}{2}$ to π, OT is positive and increases from 1 to ∞

$\therefore$ cosec θ increases from 1 to ∞.

As the angle θ passes through the value π, cosec θ suddenly changes from $+\infty$ to $-\infty$.

From π to $\frac{3\pi}{2}$, OT is negative and increases from $-\infty$ to -1.

$\therefore$ cosec θ increases from $-\infty$ to -1.

From $\frac{3\pi}{2}$ to 2π, OT is negative and decreases from -1 to $-\infty$.

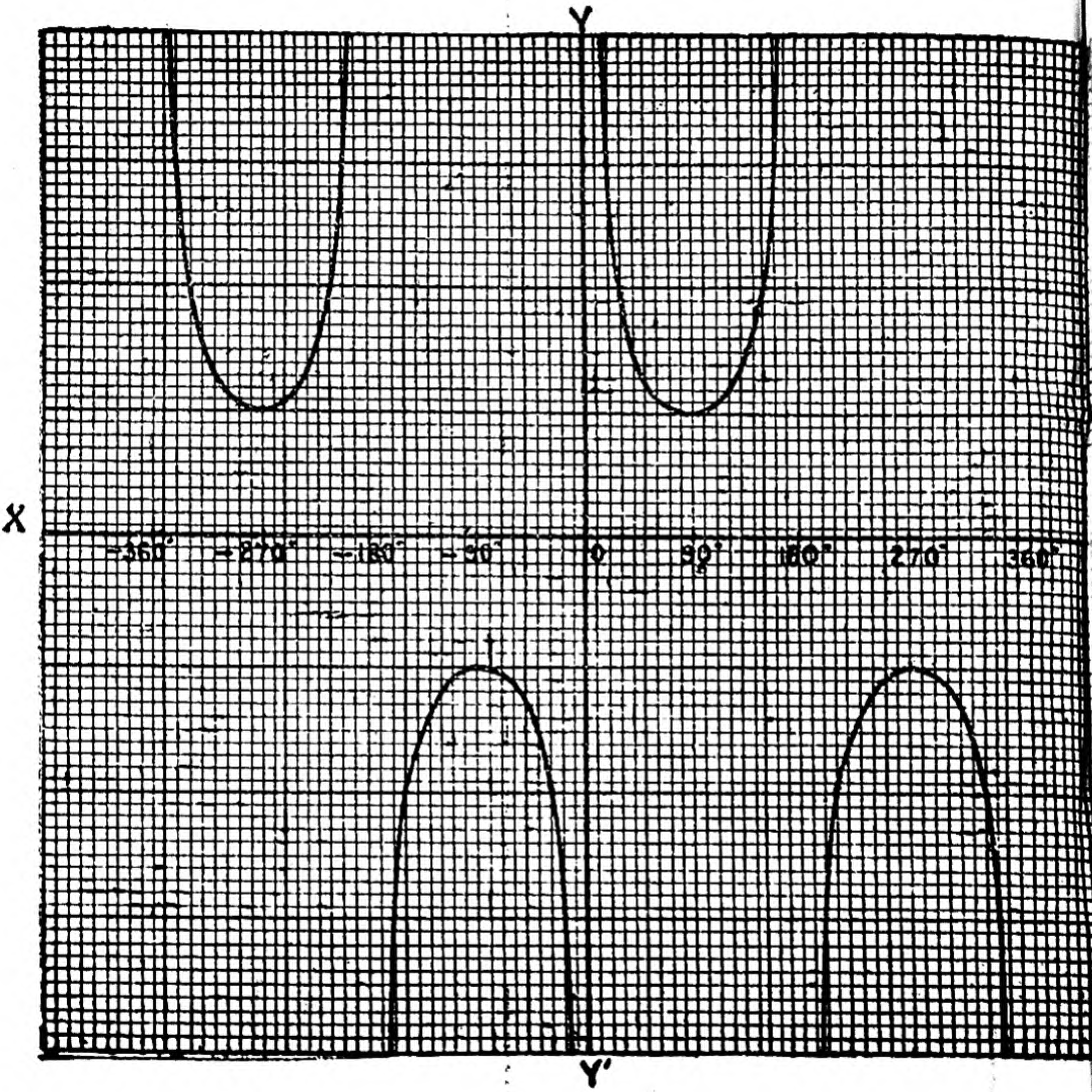
$\therefore$ cosec θ decreases from -1 to $-\infty$.

As θ passes through the value 2π, cosec θ suddenly changes from $-\infty$ to $+\infty$.

Note on Graphs. It is not within the scope of this book to go into details but it is sufficient to say that it is not necessary to have uniform scale along both the axes; in fact in many cases it is not practicable to do so. A graph drawn with different scales along the two axes serves all the purposes for which a graph is usually drawn, even though it requires a greater skill which the student is supposed to possess already.

TABLE FOR THE COSECANT GRAPH

$x=$	$-360°+0°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°-0°$	$-180°+0°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$-0°$
cosec $x=$	$+\infty$	2	1·2	1	1·2	2	$+$	$-$	-1	$-1·2$	-1	$-1·2$	-2	$-\infty$
$x=$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°-0°$	$180°+0°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°-0°$
cosec $x=$	$+$	2	1·2	1	1·2	2	$+$	$-$	-2	$-1·2$	-1	$-1·2$	-2	$-\infty$



The Cosecant Graph

Ex. 1. Show that $\sin 50° > \cos 50°$.

The angle is in the first quadrant where $\sin\theta$ increases from 0 to 1 and $\cos\theta$ decreases from 1 to 0. But at 45°, $\sin 45° = \cos 45°$ because each of them is $\frac{1}{\sqrt{2}}$. After reaching 45°, $\sin\theta$ increases while $\cos\theta$ decreases.

$\therefore \sin 50° > \cos 50°$.

Ex. 2. Determine whether $\sin A + \cos A$ is positive or negative when $A = 136°$.

The angle is in the second quadrant where $\sin A$ is positive and $\cos A$ is $-$ve. Also in this quadrant $\sin A$ decreases from 1 to 0 whereas $\cos A$ decreases from 0 to -1 and therefore $\cos A$ increases in magnitude. At 135° $\sin A$ and $\cos A$ are equal in magnitude (though opposite in sign). Therefore after that (*i. e.*, at 136°) $\cos A$ is greater than $\sin A$ in magnitude and is negative. $\therefore$ $\sin A + \cos A$ is $-$ve at $A = 136°$.

This can also be done as follows :—

$\sin 136° = \sin(180° - 44°) = \sin 44°$

$\cos 136° = \cos(180° - 44°) = -\cos 44°$.

Thus at 136°, $\sin A + \cos A = \sin 44° - \cos 44°$ But it is easy to argue, as is done in Ex. 1, that $\cos 44° > \sin 44°$.

$\therefore \sin 44° - \cos 44°$ is negative.

EXERCISE X

1. Prove that

(*i*) $\tan A - \cot A$ is positive when $A = 53°$.

(*ii*) $\sin A - \cos B$ is not negative when A and B are between 45° and 90°.

2. Prove that $\sin A + \cos A$ is positive if A lies between 45° and 135°, but negative if A is between 135° and 225°.

3. Trace the variations of $\sin\theta$ as θ varies from $-\pi$ to π and exhibit them by means of a graph. (P. U. 1942 S.)

4. Draw the graph of $y = \sin x$ as x varies from 0° to 180° and from the graph find out the values of x when (*i*) $\sin x = \cdot 3$. (*ii*) $\sin x = \cdot 6$. (P. U.)

5. Draw the graph of $y=\cos x$ when x varies from $-\pi$ to π and make use of the graph to solve the equations (*i*) $\cos x=\frac{4}{5}$. (*ii*) $\cos x=-\frac{3}{5}$.

6. With the same axes draw graphs of $y=\sin x$ and $y=\cos x$ for $0<x<2\pi$ and read off from your graph the roots of the equation $\sin x=\cos x$. (P. U.)

7. Use the graph of $y=\tan x$ to solve the equations (*i*) $\tan x=\frac{1}{2}$. (*ii*) $\tan x=-3$.

[Hint :—Here $\tan x=\frac{1}{2}$. Let $y=\tan x$ $\therefore$ $y=\frac{1}{2}$.

Thus draw the graph $y=\tan x$ and read where $y=\frac{1}{2}$ cuts it.]

8. Draw the graph of $y=\tan x$ for values of x lying between $0°$ and $180°$; show by means of this graph that $x=35$ is a solution of $x=50 \tan x$, where x is measured in degrees.

9. Trace the changes in (*i*) $\sin 2\theta$, (*ii*) $\tan 2\theta$, (*iii*) $\sec 2\theta$, as θ varies from $0°$ to $180°$ and exhibit them by means of graphs.

10. Trace the changes in $\cos \theta$ as θ varies from 0 to 2π and exhibit them graphically.

11. Taking 1 inch to represent $30°$ for x and one inch as the unit for y and plotting values of x at intervals of $30°$ between $0°$ and $180°$, draw the graphs $y=3\cos x$ and $y=\sin x$. Find to the nearest degree the value of x where these graphs intersect.

12. Draw the graphs of $y=2-3\sin x$ and $y=\sin 2x$ from $x=0$ to $180°$ using the same scales and axes. From your graphs find the approximate value of x between $0°$ and $180°$ which satisfy the equation $\sin 2x+3\sin x=2$.

13. Draw the graph of $y=3\sin x-2$ for values of x from $0°$ to $180°$. Find from the graph the angles whose sine is $\frac{2}{3}$.

14. Solve graphically the equation $3\sin x=\cos x+2$ where x is acute.

[Hint. Draw the graphs of $y=3\sin x$ and $y=2+\cos x$ with the same axes.]

15. From the graphs of $y=\sin x$ and $y=\tan x$ deduce that for $0<x<\frac{\pi}{2}$, $\sin x<x<\tan x$.

MISCELLANEOUS EXERCISE I

1. Define a unit. What is meant by saying that the measure of a quantity is n? If the unit of angular measure were 15°, what would be the measure of a rt. angle?

2. State the value in English and in French measure of—

(*i*) the sixteenth part of a right angle.

(*ii*) an interior angle of an equiangular hexagon.

3. If G, D, and θ be the number of grades, degrees and radians in any angle, prove that $G-D=\frac{20\theta}{\pi}$.

4. Define a radian and show that it is constant angle.

A horse is tethered to a stake by a rope 27 ft. long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far it will have gone when the rope has traced an angle of 70° $[\pi=\frac{22}{7}]$. (P. U. 1941).

5. The angles of a triangle are in A. P., the greatest being 105°. Find the angles in radians.

6. The angles of a triangle are in Arithmetical Progression. The number of grades in the least is to the number of radians in the greatest as 40 is to π. Find the angles in degress. (P. U. Supp. 1942.)

7. Prove that :—

$$\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}.$$

8. If $\sin\alpha=m\sin\beta$ and $\tan\alpha=n\tan\beta$,

show that $\cos^2 a=\frac{m^2-1}{n^2-1}$.

9. Prove that (*i*) $1+2(\sin^6\alpha+\cos^6\alpha)=3(\sin^4\alpha+\cos^4\alpha)$,

(*ii*) $\frac{1+\cos A}{\sec A-\tan A}\cdot\frac{1-\cos A}{\sec A+\tan A}=2(1+\tan A)$. (B. U.)

10. If $x=a \sec \theta$ and $y=b \tan \theta$, prove that

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

11. A man standing due south of a tower on a horizontal plane through its foot, finds the elevation of the top of the tower to be 60°. He goes east 8 ft. and then finds the elevation to be 30°. Find the height of the tower.

12. Complete the following equation, giving proof :

$$\tan (180°+A)=$$

Given $\cot A=\tan (n-1) A$, find one value of A,

Find the value of cos 585°, tan (−945)°, sec 1350°, and tan 570°.

13. Find the values of (*i*) sin 1110° (*ii*) cos 300° (*iii*) tan 945° (*iv*) sec 1980°.

14. Prove that $\sin (\pi+\alpha)=-\cos \alpha$, $\cos (\pi+\alpha)=-\sin\alpha$

Find the values of cos 120°, cot 510°, sec 495° and cosec 150°.

15. Prove that $\sin 600° \cos 330°+\cos 120° \sin 150°=-1.$

16. If $\tan \theta=t$, find an expression for $\cos^4\theta-\sin^4\theta$ in terms of t.

17. If $\tan \theta=t$, express $\dfrac{\tan^2\theta-\sin^2\theta}{\cot^2\theta-\cos^2\theta}$ in terms of t.

18. Show that the tangent of an angle can be expressed in terms of the tangent of an angle not greater than 45°.

19. Prove that in the general $\cos (n\pi \pm A)=(-1)^n\cos A$.

20. Prove that in general

$$\cos(2n\pi+A)=-\cos[(2n+1)\pi-A]=(\cos 2n\pi-A).$$

21. Show that

$$\frac{\sin^3(180°+\theta) \tan (360°-\theta) \sec^2 (180°-\theta)}{\cos^2 (90°-\theta) \operatorname{cosec}^2\theta \sin (180°-\theta)}=\tan^3\theta.$$

22. $\cos \theta+\sin (270°+\theta)-\sin (270°-\theta)+\cos (540°+\theta)=0.$

23. $\cot(\pi+A)+\tan(\pi+A)+\tan\left(\frac{\pi}{2}+A\right)$
$+\tan(2\pi-A)=0$.

24. $\dfrac{1}{\operatorname{cosec}(90°-A)+\cot(90°+A)}$
$=\sec A+\tan(180°+A)$.

25. Given that $\sin A=\cos(n-1)A$, find one value of A.

26. Show that each values of any circular function of an angle is, in general, repeated twice as x varies from 0 to 2π.

27. Find the values of θ lying between $0°$ and $360°$ which satisfy

(*i*) $\sin^3\theta+\cos^3\theta=0$. (*ii*) $2\sin^2\theta-5\cos\theta-4=0$.
(*iii*) $2\sin^2\theta+\sin\theta-1=0$. (*iv*) $\cos^2\theta-\sin\theta-\frac{1}{4}=0$.
(*v*) $3\sec^2\theta+5\tan^2\theta=\frac{17}{3}$.

28. Draw the graph of $y=\sin x$ for $x=0°$ to $360°$ tabulating the values of y at intervals of $15°$. Mathematical tables may be used. (C. U. 1927)

29. Sketch in one figure the graphs of $\sin 2x$ and $\tan x$ and find from your figure the solutions of the equation $\sin 2x=\tan x$. (B. U.)

30. Express each of the following in terms of an angle less than $45°$:—sin 276°, cos 183°, cot 109°, sec 222° and read tables to write down their values.

31. Show that equation $\tan\theta=1+\theta$ has an infinite number of real roots, and find graphically the approximate value of the smallest positive root. [Hint : Draw the graphs of $y=\tan\theta$ and $y=1+\theta$ with the same axes.]

32. If $\dfrac{1+\sin A}{1-\sin A}=\dfrac{a^2}{b^2}$, find $\tan A$.

33. In a quadrilateral ABCD, the angle subtended by BC, CD at A are respectively 60° and 30° ; the angles subtended by AD, DC at B are respectively 30° and 60° ; and the length of AB is 300 feet. Find the length of AC, BC, BD and AD.

34. If $\dfrac{\cos\theta}{\cos\phi}=ab$ and $\dfrac{\sin\theta}{\sin\phi}=a$, show that

$$\tan^2\theta=\frac{1-a^2b^2}{a^2-1}.$$

35. The diagonals of a quadrilateral are inclined at an angle θ and are a and b in length. Show that the area of the quadrilateral is $\frac{1}{2}ab\sin\theta$.

36. If $\tan\theta+\cot\theta=\dfrac{169}{80}$, find the value of $\sin\theta+\cos\theta$.

CHAPTER VI

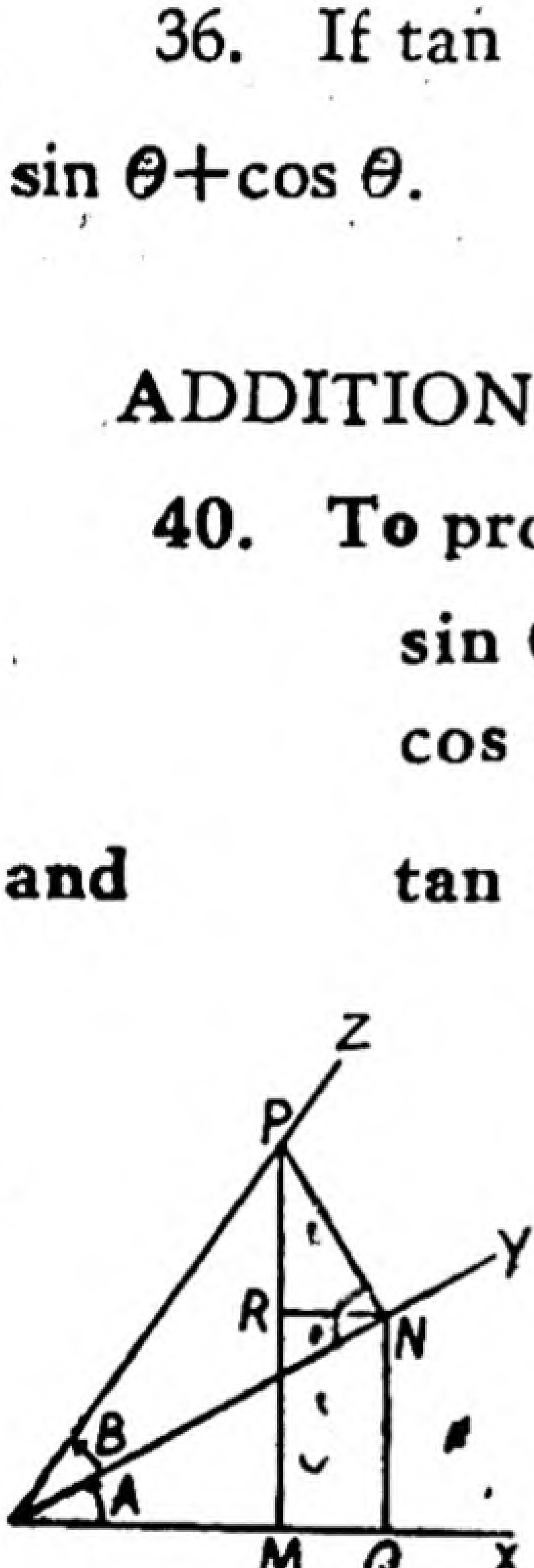
ADDITION AND SUBTRACTION FORMULAE

40. To prove that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B;$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B;$$

and $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$



Let the revolving line starting from the initial position OX, trace out an angle XOY (=A) and then furthur trace out an angle YOZ (=B), so that angle XOZ=A+B.

From any point P in the final position OZ of the revolving line draw PM and PN perpendiculars to OX and OY; from N, draw NQ perpendicular to OX and NR perpendicular to MP.

Then $\quad \angle RPN=90^\circ-\angle RNP=\angle RNO=\angle NOQ=A.$

Hence $\sin(A+B)=\sin XOZ=\dfrac{MP}{OP}=\dfrac{MR+RP}{OP}$

$$=\frac{QN+RP}{OP}=\frac{QN}{OP}+\frac{RP}{OP}$$

$$=\frac{QN.ON}{ON.OP}+\frac{RP.NP}{NP.OP}$$

$$=\sin A \cos B+\cos \angle RPN \sin B$$

$$=\mathbf{\sin A \cos B+\cos A \sin B.}$$

Again, $\cos (A+B)=\cos \angle XOZ=\frac{OM}{OP}=\frac{OQ-MQ}{OP}$

$$=\frac{OQ}{OP}-\frac{RN}{OP}=\frac{OQ}{ON}\cdot\frac{ON}{OP}-\frac{RN}{NP}\cdot\frac{NP}{OP}$$

$$=\cos A \cos B-\sin \angle RPN \sin B$$

$$=\mathbf{\cos A \cos B-\sin A \sin B}$$

$$\tan (A+B)=\tan XOZ==\frac{MP}{OM}=\frac{MR+RP}{OQ-MQ}$$

$$=\frac{QN+RP}{OQ-RN}=\frac{\frac{QN}{OQ}+\frac{RP}{OQ}}{1-\frac{RN}{OQ}}=\frac{\tan A+\frac{RP}{OQ}}{1-\frac{RN.RP}{RP.OQ}}$$

But $\frac{RN}{RP}=\tan A$, and from the similar triangles RPN and QON, we have $\frac{RP}{OQ}=\frac{NP}{ON}=\tan B.$

Hence $\quad \mathbf{\tan (A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B}}.$

Another Method for showing that

$$\tan (A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B}.$$

$$\tan (A+B)=\frac{\sin (A+B)}{\cos (A+B)}$$

$$=\frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B-\sin A \sin B}$$

$$=\frac{\frac{\sin A \cos B}{\cos A \cos B}+\frac{\cos A \sin B}{\cos A \cos B}}{1+\frac{\sin A \sin B}{\cos A \cos B}}=\frac{\tan A+\tan B}{1-\tan A \tan B}.$$

Note 1.—The figure in the proof is drawn out for the case in which A, B and A+B are all acute angles. The same proof will apply to angles of any size, due attention being paid to the signs of the

quantities involved (See Chapter XVI).

However the following method of proof may also be adopted.

Case I. Let A, B, $A+B$ be all acute angles, then the results follow as in Art. 40.

Case II. Let one of the two component angles A_1, B_1 say A_1 be obtuse,

i.e., $A_1=90°+A$ where $A+B_1$ is acute.

Then $\sin(A_1+B_1)=\sin(90°+A+B_1)$

$$=\cos(A+B_1)$$
$$=\cos A\cos B_1-\sin A\sin B_1$$
$$=\cos(A_1-90°)\cos B_1-\sin(A_1-90°)\sin B_1$$
$$=\sin A_1\cos B_1+\cos A_1\sin B_1$$

Also $\cos(A_1+B_1)=\cos(90°+A+B_1)$

$$=-\sin(A+B_1)$$
$$=-\sin A\cos B_1-\cos A\sin B_1$$
$$=-\sin(A_1-90°)\cos B_1-\cos(A_1-90°)\sin B_1$$
$$=\cos A_1\cos B_1-\sin A_1\sin B_1.$$

It follows from these two that formula for $\tan(A_1+B_1)$ must also hold.

Similarly the case where A_1, B_1 both become obtuse can be dealt with.

Thus the formulae of Art. 40 are true for component angles lying between 0° and 180°.

Case III. If the component angles A_2 B_2 be between 0° and 270° we are only to put $A_2=90°+A_1$ or $B_2=90°+B_1$ and then this case also follows from Case II in exactly the same manner as Case II was derived from Case I.

By proceeding in this way we see that the theorems are true universally.

Caution :—*Notice that a circular function of $A+B$ is not equal to the sum of the corresponding circular function of A and B ; thus $\sin(A+B)\neq\sin A+\sin B$.*

Note 2. In the above formulae cot (A+B), sec (A+B) and cosec (A+B) follow from tan (A+B), cos (A+B) and sin (A+B) respectively. For example.

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-\dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B}+\dfrac{\cos A\sin B}{\sin A\sin B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}.$$

Similarly it may be proved that

$$\sec(A+B)=\frac{\sec A\sec B\operatorname{cosec}A\operatorname{cosec}B}{\operatorname{cosec}A\operatorname{cosec}B-\sec A\sec B}$$

$$\operatorname{cosec}(A+B)=\frac{\sec A\sec B\operatorname{cosec}A\operatorname{cosec}B}{\sec A\operatorname{cosec}B-\operatorname{cosec}A\sec B}.$$

Ex. 1. Find the values of sin 75°, cos 75°, tan 75°.

$$\sin 75^\circ=\sin(45^\circ+30^\circ)$$

$$=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2}+\frac{1}{\sqrt2}\cdot\frac{1}{2}=\frac{\sqrt3+1}{2\sqrt2}.$$

Similarly $\cos 75^\circ=\cos(45^\circ+30^\circ)=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2}-\frac{1}{\sqrt2}\cdot\frac{1}{2}=\frac{\sqrt3-1}{2\sqrt2},$$

$$\tan 75^\circ=\frac{\sin 75^\circ}{\cos 75^\circ}=\frac{\sqrt3+1}{\sqrt3-1}.$$

Or $\tan 75^\circ=\tan(45^\circ+30^\circ)=\dfrac{\tan 45^\circ+\tan 30^\circ}{1-\tan 30^\circ\tan 45^\circ}$

$$=\frac{1+\dfrac{1}{\sqrt3}}{1-\dfrac{1}{\sqrt3}}=\frac{\sqrt3+1}{\sqrt3-1}.$$

Ex. 2. Find R and θ when $R\sqrt3\sin(30^\circ+\theta)=15$, $R\sin\theta=5$, θ being acute.

Dividing one equation by the other we get

$$\sqrt3\,\frac{\sin(30^\circ+\theta)}{\sin\theta}=3,$$

or $\dfrac{\sin 30^\circ \cos\theta + \cos 30^\circ \sin\theta}{\sin\theta} = \sqrt{3}$

or $\frac{1}{2}\cot\theta + \dfrac{\sqrt{3}}{2} = \sqrt{3}$

$\therefore$ $\cot\theta = \sqrt{3}$

or $\theta = 30^\circ$.

From $R\sin\theta = 5$, we get $R = 10$.

EXERCISE XI

Prove that

1. $\cos(A+45^\circ) = \dfrac{1}{\sqrt{2}}(\cos A - \sin A)$.

2. $\sin(45^\circ + A) = \dfrac{1}{\sqrt{2}}(\cos A + \sin A)$.

3. Show that $\tan(45^\circ + \theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$.

4. Show that $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$.

5. $\dfrac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} = 1 - \tan\alpha\tan\beta$

6. $\sin\alpha + \sin\left(\alpha + \dfrac{2\pi}{3}\right) + \sin\left(\alpha + \dfrac{4\pi}{3}\right) = 0$ for all values of α.

7. $\cos(60^\circ + \alpha) + \sin(30^\circ + \alpha) = \cos\alpha$.

8. If $\sin A = \frac{4}{5}$, and $\sin B = \frac{5}{13}$, find the value of $\sin(A+B)$, when A and B are both acute.

9. If $\sin A = \dfrac{1}{\sqrt{10}}$ and $\sin B = \dfrac{1}{\sqrt{\text{[illegible]}}}$, show that $A+B = 45^\circ$.

10. If $\sin A = \dfrac{2ab}{a^2+b^2}$ and $\sin B = \dfrac{2cd}{c^2+d^2}$, find $\sin(A+B)$.

11. If $A+B+C+D = 180^\circ$, prove that
$\cos A\cos B + \cos C\cos D = \sin A\sin B + \sin C\sin D$.

12. The cosines of two angles of a triangle are $\frac{4}{5}$ and $\frac{12}{13}$ respectively; find the cosine of the third angle.

13. Find the value of $\sin 22^\circ \cos 38^\circ + \cos 22^\circ \sin 38^\circ$.

14. Given that $\tan \alpha = \frac{3}{4}$ and $\sin \beta = \frac{5}{13}$ α, β being both positive acute angles, find the value of $\tan(\alpha+\beta)$.

Simplify the following, reducing it to a single term :

15. $\sin 2x \cos 3x + \cos 2x \sin 3x$.

16. $\cos 3x \cos 5x - \sin 3x \sin 5x$.

17. Show that

$\cos(A-B)\cos B - \sin(A+B)\sin B$ is independent of B.

18. Show that $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

[*Hint.* $A-B = A+(-B)$

$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$, etc.]

41. Trigonometrical Ratios of Multiple Angles.

(a) *Circular functions of an angle in terms of half the angle* :—

$$\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A \quad \ldots\ldots\ldots (\textit{First Form})$$

$$= \frac{2 \sin A \cos A}{\sin^2 A + \cos^2 A} = \frac{\dfrac{2 \sin A \cos A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}}$$

$$= \frac{2 \tan A}{1+\tan^2 A} \quad \ldots\ldots\ldots\ldots (\textit{Second Form})$$

$$\cos 2A = \cos(A+A) = \cos A . \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A \quad \ldots\ldots\ldots\ldots (\textit{First Form})$$
$$= 1 - \sin^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A \quad \ldots\ldots\ldots\ldots (\textit{Second Form})$$

Again $$= \cos^2 A - (1 - \cos^2 A)$$
$$= 2\cos^2 A - 1 \quad \ldots\ldots\ldots\ldots (\textit{Third Form})$$

Again $$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}}$$

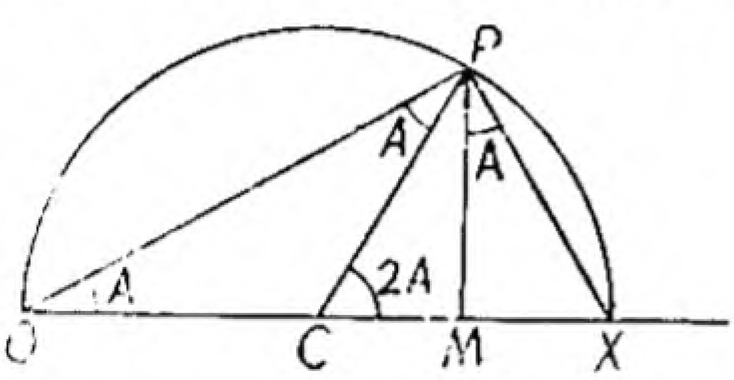
$$= \frac{1-\tan^2 A}{1+\tan^2 A} \quad \ldots\ldots\ldots\ldots (\textit{Fourth Form})$$

$$\tan 2A = \tan(A+A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}.$$

Note.—A geometrical method of proving the first three results may be obtained by making $B = A$ in article 40 and going through the same proof with this change.

These results may be proved geometrically in the following manner also.

Take $\angle XOP = A$. With any point C on OX as centre and radius equal to CO ($= a$, say) draw a circle cutting OP and OX in P and X respectively. Join OP, PX and draw $PM \perp OX$. Then



$\angle CPO = \angle COP = A$
$\angle XCP = \angle CPO + \angle COP = 2A$
and $\angle XPM = 90° - \angle PXM$
$= \angle COP = A.$
Now $MP = PC \sin PCM$
$= a \sin 2A.$
Also $MP = OX \frac{OP}{OX} \cdot \frac{MP}{OP}$
$= 2a \cos A \sin A.$
$\therefore a \sin 2A = 2a \sin A \cos A.$
i. e., $\sin 2A = 2 \sin A \cos A.$
Again $OM = OC + CM$ and
$MX = CX - CM$
$\therefore OM - MX2 = CM$ ...(i)

also $OM = OX \frac{OP}{OX} \cdot \frac{OM}{OP} = 2a \cos A \cos A = 2a \cos^2 A.$

and $MX = OX \frac{PX}{OX} \cdot \frac{MX}{PX} = 2a \sin A \sin A = 2a \sin^2 A$

Also $CM = CP \cos 2A = a \cos 2A$
Substituting in (*i*), we get
$2a \cos 2A = 2a \cos^2 A - 2a \sin^2 A$
i. e, $\cos 2A = \cos^2 A - \sin^2 A$
Again, $CM = OM - OC = OM - a$ .....(ii)
and also $= CX - MX = a - MX$ ......(iii)
Substituting in (*ii*), we get $a \cos 2A = 2a \cos^2 A - a$
i. e., $\cos 2A = 2\cos^2 A - 1$
Substituting in (*iii*), we get $a \cos 2A = a - 2a \sin^2 A$
e., $\cos 2A = 1 - \sin^2 A$.

$$\text{Again } \tan 2A = \frac{MP}{CM} = \frac{2\,MP}{2CM} = \frac{2MP}{OM - MX} = \frac{2\frac{MP}{OM}}{1 - \frac{MX}{MP} \cdot \frac{MP}{OM}}$$
$$= \frac{2 \tan A}{1 - \tan^2 A}.$$

(*b*) To prove that

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

and **(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.**

(*i*) $\sin 3A = \sin(2A + A)$

$$\begin{aligned}
&= \sin 2A \cos A + \cos 2A \sin A \\
&= 2\sin A \cos A \cos A + (1 - 2\sin^2 A)\sin A \\
&= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^2 A \\
&= 3\sin A - 4\sin^3 A.
\end{aligned}$$

(*ii*) $\cos 3A = \cos(2A + A)$

$$\begin{aligned}
&= \cos 2A \cos A - \sin 2A \sin A \\
&= (2\cos^2 A - 1)\cos A - 2\sin A \cos A \sin A \\
&= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A) \\
&= 4\cos^3 A - 3\cos A.
\end{aligned}$$

(*iii*) $\tan 3A = \tan(2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A} \cdot \tan A}$$

$$= \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Otherwise thus :

$$\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}$$

$$= \frac{3\dfrac{\sin A}{\cos^3 A} - 4\dfrac{\sin^3 A}{\cos^3 A}}{4 - 3\dfrac{\cos A}{\cos^3 A}} = \frac{3\tan A \sec^2 A - 4\tan^3 A}{4 - 3\sec^2 A}$$

$$= \frac{3\tan A(1 + \tan^2 A) - 4\tan^3 A}{4 - 3(1 + \tan^2 A)}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Ex. 1. Show that $\tan\theta=\dfrac{\sin 2\theta}{1+\cos 2\theta}=\dfrac{1-\cos 2\theta}{\sin 2\theta}$.

Here $\dfrac{\sin 2\theta}{1+\cos 2\theta}=\dfrac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1}=\tan\theta$

$$\frac{1-\cos 2\theta}{\sin 2\theta}=\frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}=\tan\theta.$$

Ex. 2. Show that $\cot\theta=\sqrt{\dfrac{1+\cos 2\theta}{1-\cos 2\theta}}$

$$\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}=\sqrt{\frac{1+2\cos^2\theta-1}{1-(1-2\sin^2\theta)}}=\frac{\cos\theta}{\sin\theta}=\cot\theta.$$

Ex. 3. Eliminate A between the equations
$\cos A+\sin A=p$, $\cos 2A=q$.

$\cos 2A=\cos^2 A-\sin^2 A=(\cos A+\sin A)(\cos A-\sin A)=q$
Putting $\cos A+\sin A=p$ in this, we get

$$\cos A-\sin A=\frac{q}{p}.$$

Adding and subtracting the last two equations, we get

$$\cos A=\frac{1}{2}\left(p+\frac{q}{p}\right)$$

$$\sin A=\frac{1}{2}\left(p-\frac{q}{p}\right).$$

Squaring and adding, we get

$$1=\frac{1}{4}\left(2p^2+2\frac{q^2}{p^2}\right)$$

or

$$2=p^2+\frac{q^2}{p^2}.$$

Otherwise thus :—We have

$$\cos A+\sin A=p$$

$$\text{and } \cos A-\sin A=\frac{q}{p} \text{ (as above)}$$

Square and add ; we get

$$2=p^2+\frac{q^2}{p^2}$$

EXERCISE XII

1. If $\cos A=\frac{3}{5}$, find $\cos 2A$.
2. If $\sin A=\frac{1}{7}$, find $\cos 2A$.
3. If $\sin A=\frac{12}{13}$ find $\sin 2A$.
4. If $\tan\theta=5$, find $\tan 2\theta$.
5. If $\tan\theta=2$, find $\sin 2\theta$ and $\cos 2\theta$.
6. Find the value of $2\sin 22\frac{1}{2}^\circ\cos 22\frac{1}{2}^\circ$.
7. Find the value of $1-2\sin^2 15^\circ$.

Prove that

8. $\dfrac{1-\cos 2\theta}{\sin 2\theta}=\tan\theta$. 9. $1-\sin 2\theta=(\sin\theta-\cos\theta)^2$.

10. $1+\sec 2\theta=\dfrac{\tan 2\theta}{\tan\theta}$. 11. $\dfrac{1+\cos A}{\sin A}=\cot\dfrac{A}{2}$.

12. $\cos 2A-\sin 2A=(\cos A-\sin A)^2-2\sin^2 A$.

13. $\cos^4\theta-\sin^4\theta=\cos 2\theta$. 14. $\cot\theta-\tan\theta=2\cot 2\theta$.

15. $\dfrac{\cot A-\tan A}{\cot A+\tan A}=\cos 2A$.

16. $\left(\sin\dfrac{A}{2}+\cos\dfrac{A}{2}\right)^2=1+\sin A$.

17. $2\sin^3 A+\sin 2A\cos A=2\sin A$.

18. $2\cos^3 A-\cos 2A\cos A=\cos A$.

19. $\dfrac{1-\cos 2\theta}{1+\cos 2\theta}=\tan^2\theta$.

Prove that

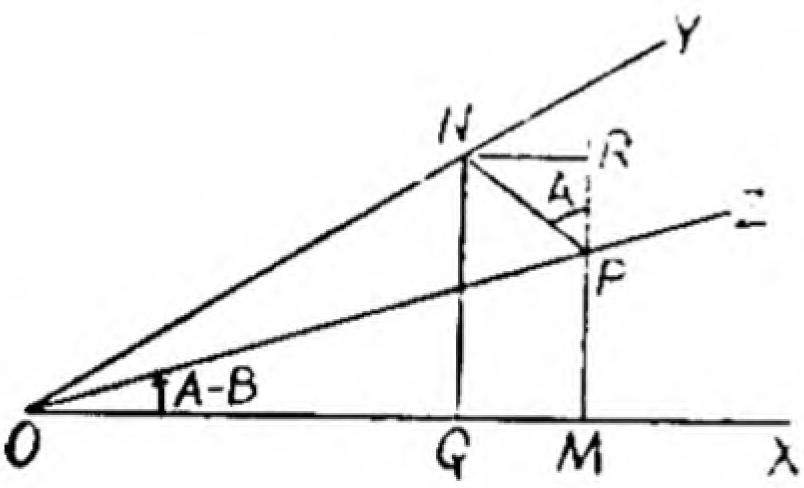
20. $8\sin^4 A=\cos 4A-4\cos 2A+3$.
21. $8\cos^4 A=\cos 4A+4\cos 2A+3$.
22. $\tan A+\tan(60^\circ+A)+\tan(120^\circ+A)=3\tan 3A$.
23. $\sin^3\theta+\sin^3(120^\circ+\theta)+\sin^3(240^\circ+\theta)=-\frac{3}{4}\sin 3\theta$.
24. $4\sin A\sin\left(A-\dfrac{\pi}{3}\right)\sin\left(A-\dfrac{2\pi}{3}\right)=\sin 3A$.
25. $\tan^3\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=\dfrac{1-\sin\theta}{1+\sin\theta}\cdot\dfrac{\cos\theta}{1+\sin\theta}$.
26. $(3\sin A-\sin 3A)^{\frac{2}{3}}+(3\cos A+\cos 3A)^{\frac{2}{3}}=4^{\frac{2}{3}}$.

27. If $\tan\theta=\frac{b}{a}$, prove that $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}$
$$=\frac{2\cos A}{\sqrt{\cos 2A}}.$$

28. If $x+\frac{1}{x}=2\cos\theta$, prove that $x^3+\frac{1}{x^3}=2\cos 3\theta$.

29. If $\tan\theta=\frac{b}{a}$ prove that $a\cos 2\theta+b\sin 2\theta=a$.

41. To prove that
$\sin(A-B)=\sin A\cos B-\cos A\sin B$;
$\cos(A-B)=\cos A\cos B+\sin A\sin B$;
and $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$.



Let the revolving line, starting from the initial position OX, trace out an angle XOY $(=A)$ and then trace back an angle YOZ equal to B in magnitude, so that the angle $XOZ=A-B$.

From any point P in the final position OZ of the revolving line, draw PM and PN perpendiculars to OX and OY respectively; from N draw NQ perpendicular to OX and NR perpendicular to MP.

Then $\angle RPN=90^\circ-\angle RNP=\angle RNY=\angle YOX=A$.

Hence $\sin(A-B)=\frac{MP}{OP}=\frac{MR-PR}{OP}$

$$=\frac{QN-PR}{OP}=\frac{QN}{OP}-\frac{PR}{OP}=\frac{QN}{ON}\cdot\frac{ON}{OP}-\frac{PR}{NP}\cdot\frac{NP}{OP}$$

$=\sin A\cos B-\cos\angle RPN\sin B$

$=\sin A\cos B-\cos A\sin B$.

Again, $\cos(A-B)=\frac{OM}{OP}=\frac{OQ+QM}{OP}=\frac{OQ+NR}{OP}$

$$=\frac{OQ}{OP}+\frac{NR}{OP}=\frac{OQ}{ON}\cdot\frac{ON}{OP}+\frac{NR}{NP}\cdot\frac{NP}{OP}$$

$$= \cos A \cos B + \sin \angle RPN \sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

$$\tan(A-B) = \frac{MP}{OM} = \frac{MR-PR}{OQ+QM} = \frac{QN-PR}{OQ+NR}$$

$$= \frac{\dfrac{QN}{OQ} - \dfrac{PR}{OQ}}{1+\dfrac{NR}{OQ}} = \frac{\tan A - \dfrac{PR}{OQ}}{1+\dfrac{NR}{PR}\cdot\dfrac{PR}{OQ}}$$

But $\frac{NR}{PR} = \tan A$; and from similar triangles RPN and QON, we have $\frac{PR}{OQ} = \frac{PN}{ON} = \tan B$.

Hence $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$.

Caution. *Notice that a circular function of A—B is not equal to the difference of the corresponding circular functions of A and B. Thus sin (A - B) is not equal to sin A - sin B.*

Another Method to show that

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} - \dfrac{\cos A \sin B}{\cos A \cos B}}{1+\dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

Cor. $\tan\left(\frac{\pi}{4} - A\right) = \dfrac{\tan\frac{\pi}{4} - \tan A}{1+\tan\frac{\pi}{4}\tan A} = \dfrac{1-\tan A}{1+\tan A}.$

Also $\cot(A-B) = \dfrac{\cos(A-B)}{\sin(A-B)}$

$$=\frac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$$

$$=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}+1}{\dfrac{\cos B}{\sin B}-\dfrac{\cos A}{\sin A}}=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

While proving the addition and subtraction formulæ, we have drawn the figure for the case when A, B, and A+B and A−B are acute angles.

But the above method of proof is applicable to all cases regard being had to the signs of the various quantities involved. It would form a good exercise for the student to go through the construction and the proof in the different cases.

For another method of proving the addition and the subtraction formulæ, see Chapter XV.

Note.—The same method of proof as in Note 1, Art. 40 may also be followed in this case.

Ex. Find sin 15°, cos 15°, and tan 15°.

$$\sin 15^\circ=\sin(45^\circ-30^\circ)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\cos 15^\circ=\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\tan 15^\circ=\frac{\sin 15^\circ}{\cos 15^\circ}=\frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

It may be observed that $15^\circ=60^\circ-45^\circ$ will give the same result.

43. To prove that

$\sin(A+B)\sin(A-B)=\sin^2A-\sin^2B.$

We have $\sin(A+B)=\sin A\cos B+\cos A\sin B,$

and $\sin(A-B)=\sin A\cos B-\cos A\sin B.$

Multiplying these two equations we get

$\sin(A+B)\sin(A-B)=\sin^2A\cos^2B-\cos^2A\sin^2B$

$=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$

$=\mathbf{\sin^2 A-\sin^2 B}.$

44. To prove that

$$\mathbf{\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A}.$$

We have $\cos(A+B)=\cos A\cos B-\sin A\sin B$

and $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

Multiplying these two equations, we get

$\mathbf{\cos(A+B)\cos(A-B)}=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$

$=\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$

$=\mathbf{\cos^2 A-\sin^2 B}$

$=(1-\sin^2 A)-(1-\cos^2 B)=\mathbf{\cos^2 B-\sin^2 A},$

Ex. 1. Simplify :

$\sin(45^\circ-x)\cos(45^\circ-y)+\cos(45^\circ-x)\sin(45^\circ-y).$

Let $45^\circ-x=A$ and $45^\circ-y=B$. The given expression therefore $=\sin A\cos B+\cos A\sin B=\sin(A+B)$

$=\sin(90^\circ-\overline{x+y})=\cos(x+y).$

Ex. 2. Show that $\dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta.$

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta.$$

Ex. 3. If $\tan B=\dfrac{n\sin A\cos A}{1-n\sin^2 A}$,

prove that $\tan(A-B)=(1-n)\tan A.$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$=\frac{\tan A-\dfrac{n\sin A\cos A}{1-n\sin^2 A}}{1+\tan A\dfrac{n\sin A\cos A}{1-n\sin^2 A}}$$

$$=\frac{\tan A-n\sin^2 A\tan A-n\sin A\cos A}{1-n\sin^2 A+n\sin A\cos A\tan A}$$

$=\tan A-n\sin^2 A\tan A-n\sin A\cos A$

$=\tan A-n(1-\cos^2 A)\tan A-n\sin A\cos A$

$=\tan A-n\tan A+n\cos^2 A\tan A-n\sin A\cos A$

$=(1-n)\tan A - n\cos A\sin A + n\sin A\cos A$

$=(1-n)\tan A.$

Ex. 4. Prove that

$\tan 11A - \tan 4A - \tan 7A = \tan 11A \tan 4A \tan 7A.$

$$\tan 11A = \tan (7A+4A) = \frac{\tan 7A + \tan 4A}{1 - \tan 7A \tan 4A}$$

$\therefore \tan 11A - \tan 7A \tan 4A \tan 11A = \tan 7A + \tan 4A$

Hence $\tan 11A - \tan 4A - \tan 7A = \tan 11A \tan 4A \tan 7A.$

EXERCISE XIII

1. Prove that $\sin(30° - A) = \frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A.$

2. If $\sin A = \frac{3}{5}$ and $\sin B = \frac{4}{5}$ find $\cos (A-B)$ and $\sin (A-B)$ where A and B are both acute.

Prove that

3. $\dfrac{\sqrt{3}\cos 23° - \sin 23°}{2} = \cos 53°.$

4. $\dfrac{\sin (\alpha-\beta)}{\sin \alpha \sin \beta} = \cot \beta - \cot \alpha.$

5. $\dfrac{\sin (\alpha-\beta)}{\sin \alpha \sin \beta} + \dfrac{\sin (\beta-\gamma)}{\sin \beta \sin \gamma} + \dfrac{\sin (\gamma-\alpha)}{\sin \gamma \sin \alpha} = 0.$

6. $\dfrac{\cos (A+B)}{\cos (A-B)} = \dfrac{1-\tan A \tan B}{1+\tan A \tan B}.$

7. $\cos (A+B) + \sin (A-B) = (\cos A + \sin A) \times (\cos B - \sin B).$

Simplify into a single term :

8. $\cos (A+B) \cos B + \sin (A+B) \sin B.$

9. $\sin (x+y) \cos x - \cos (x+y) \sin x.$

10. Prove that—

(*i*) $\sin (60°+\theta) - \sin (60°-\theta) = \sin \theta.$

(*ii*) $\sin (2n+1)\theta \sin \theta = \sin^2(n+1)\theta - \sin^2 n\theta.$

11. ABC is an isosceles triangle, right-angled at C ; D is the middle point of AC. prove that DB divides the angle B into two parts whose cotangents are 2 and 3.

12. If $\dfrac{P}{W} = \dfrac{\sin (\alpha+\phi)}{\cos \phi}$; and $\tan \phi = \mu$, find a simple expression for P

13. In a certain problem $R=ut-\frac{1}{2}gt^2\sin A$. Find a simpler expression for R, if $u=v\cos\theta$ and $t=\dfrac{2v\sin\theta}{g\cos A}$.

14. Prove that $\sin^2 A+\sin^2 B=1-\cos(A-B)\cos(A+B)$.

15. $\tan^2 A-\tan^2 B=\dfrac{\sin(A+B)\sin(A-B)}{\cos^2 A\cos^2 B}$.

16. $(\cos A-\cos B)^2+(\sin A+\sin B)^2=4\sin\dfrac{2A+B}{2}$.

17. $(\cos A-\cos B)^2+(\sin A-\sin B)^2=4\sin^2\dfrac{A-B}{2}$.

18. $\cos^4 A-\cos^4 B=\sin(A+B)\sin(B-A)\times\{1+\cos(A+B)\cos(A-B)\}$.

19. From the equation $\sin(\alpha+\beta)\cos\theta=2\sin\alpha\cos(\beta-\theta)$ show that $\tan\theta=\dfrac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}$.

45. By repeated applications of Articles 40 and 42 the trigonometrical ratios of the sum or difference of more than two angles can be easily obtained.

Thus

$$\begin{aligned}
\sin(A+B+C)&=\sin[(A+B)+C]\\
&=\sin(A+B)\cos C+\cos(A+B)\sin C\\
&=(\sin A\cos B+\cos A\sin B)\cos C\\
&\quad+(\cos A\cos B-\sin A\sin B)\sin C\\
&=\sin A\cos B\cos C+\sin B\cos C\cos A\\
&\quad+\sin C\cos A\cos B-\sin A\sin B\sin C.
\end{aligned}$$

$$\begin{aligned}
\cos(A+B+C)&=\cos[(A+B)+C]\\
&=\cos(A+B)\cos C-\sin(A+B)\sin C\\
&=(\cos A\cos B-\sin A\sin B)\cos C\\
&\quad-(\sin A\cos B+\cos A\sin B)\sin C.\\
&=\cos A\cos B\cos C-\cos A\sin B\sin C\\
&\quad-\cos B\sin C\sin A-\cos C\sin A\sin B.
\end{aligned}$$

$$\begin{aligned}
\tan(A+B+C)&=\tan[(A+B)+C]\\
&=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}\\
&=\frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1+\dfrac{\tan A+\tan B}{1-\tan A\tan B}\tan C}
\end{aligned}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

Otherwise thus : $\tan(A+B+C)=\dfrac{\sin(A+B+C)}{\cos(A+B+C)}$

$= (\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B$
$-\sin A\sin B\sin C)$

$\div(\cos A\cos B\cos C-\cos A\sin B\sin C-\cos B\sin C\sin A$
$-\cos C\sin A\sin B).$

Dividing numerator and denominator by

$\cos A\cos B\cos C$, we get

$\tan(A+B+C)$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}.$$

Cor. 1. If $A+B+C=\pi$, then $\tan(A+B+C)=0$ and therefore $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

Another Method.

$A+B+C=\pi$

$\therefore\ A+B=\pi-C$

$\therefore\ \tan(A+B)=\tan(\pi-C)$

$\therefore\ \dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C.$

$\therefore\ \tan A+\tan B=-\tan C+\tan A\tan B\tan C.$

Hence $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

Note.—The first method is to be preferred.

Cor. 2. If $A+B+C=\dfrac{\pi}{2}$, $\tan\dfrac{\pi}{2}\to\infty$

$$\therefore\ \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}\to\infty$$

$\therefore\ 1-\tan A\tan B-\tan B\tan C-\tan C\tan A=0.$

Hence $\tan A\tan B+\tan B\tan C+\tan C\tan A=1.$

Another Method.

$A+B+C=\dfrac{\pi}{2}$ $\qquad\qquad \therefore\ A+B=\dfrac{\pi}{2}-C$

$\therefore\ \tan\ (A+B)=\tan\left(\frac{\pi}{2}-C\right)=\cot C$

or $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{1}{\tan C}$

or $\tan A\tan C+\tan B\tan C=1-\tan A\tan B$

or $\tan A\tan B+\tan B\tan C+\tan C\tan A=1.$

EXERCISE XIV

1. Deduce from Art. 45 formulae for sin 3A, cos 3A, tan 3A.

2. Write down: $\sin (A+B-C)$; $\cos (A-B-C)$; $\tan (A+B-C)$.

3. Prove that $\tan (A-B)\tan (B-C)\tan (C-A) =\tan (A-B)+\tan (B-C)+\tan (C-A)$.

4. If $A+B+C=180^\circ$; show that (*i*) $\cos A\cos B\cos C =\cos A\sin B\sin C+\cos B\sin C\sin A+\cos C\sin A\sin B$.
(*ii*) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.
(*iii*) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

5. Show that $\dfrac{\sin 4A}{\cos 2A}+\dfrac{\cos 4A}{\sin 2A}=\operatorname{cosec} 2A.$

6. Find $\tan (A+B+C)$ and $\tan (A+B+C+D)$ in terms of S_1, S_2, S_3, S_4, where S_1, S_2, S_3, S_4 denotes the sums of tangents of the angles A, B, C, D taken one, two, three and four at a time respectively.

46. Some Important Solved Questions.

Ex. 1. It is often required to change the binomial or the two term expression $a\sin A+b\cos A$ into an expression of the form $r\sin (A+B)$ or $r\cos (A+B)$, where B is an angle.

Let us put $a\sin A+b\cos A=r\sin (A+B)$
$=r(\sin A\cos B+\cos A\sin B)$
$=r\sin A\cos B+r\cos A\sin B.$

Equating coefficients of sin A and cos A on both the sides, we have $r\cos B=a$
and $r\sin B=b.$

Squaring and adding these last two equations, we get
$r^2(\cos^2 B+\sin^2 B)=a^2+b^2$ or $r=\sqrt{a^2+b^2}.$

This gives r. Having known r, B is determined without ambiguity by the equations $\cos B = \frac{a}{r}$ and $\sin B = \frac{b}{r}$,

i.e., $\cos B = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin B = \frac{b}{\sqrt{a^2+b^2}}$.

It may be remarked that it is customary to take $r = +\sqrt{a^2+b^2}$, that is, r is generally taken to be positive. We are thus led to the following method.

In order to put $a \sin A + b \cos A$ in the form $r \sin(A+B)$, we proceed as follows:

Put $a = r \cos B$
$b = r \sin B$.

quaring and adding, $r^2 = a^2+b^2$ or $r = \sqrt{a^2+b^2}$ and putting this value of r in these equations we get

$$\cos B = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin B = \frac{b}{\sqrt{a^2+b^2}}.$$

Thus $a \sin A + b \cos A = \sqrt{a^2+b^2} \sin(A+B)$, where B is an angle given by $\cos B = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin B = \frac{b}{\sqrt{a^2+b^2}}$.

Similarly in order to put $a \sin A + b \cos A$ in the form $r \cos(A+B)$, we proceed thus:

Put $a = r \sin B$
$b = r \cos B$.

Squaring and adding these, we get $r^2 = a^2+b^2$ or $r = \sqrt{a^2+b^2}$ and $\sin B = \frac{a}{r} = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos B = \frac{b}{\sqrt{a^2+b^2}}$.

Thus B is found without ambiguity.

$\therefore\ a \sin A + b \cos A = r \sin B \sin A + r \cos B \cos A$

$$= r \cos(A-B).$$
$$= \sqrt{a^2+b^2} \cos(A-B).$$

Ex. 2. Show that $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$.

Here put $1 = r \cos B$ and $1 = r \sin B$. Squaring and adding we get $r^2 = 2$ or $r = \sqrt{2}$.

and $\cos B=\frac{1}{\sqrt{2}}$ and $\sin B=\frac{1}{\sqrt{2}}$ so that $B=45^\circ$.

Thus $\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$.

Ex. 3. Trace the changes in the value of $\sin\theta+\cos\theta$ as θ varies from -45° to 315° and draw its graph.

In this case $\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$.

Case . When $\theta=-\frac{\pi}{4}$, $\sin\theta+\cos\theta=0$.

As θ increases, i.e., decreases in magnitude the expression goes on increasing because $\sin\left(\theta+\frac{\pi}{4}\right)$ increases; when $\theta=\frac{\pi}{4}$ then $\sin\theta+\cos\theta=\sqrt{2}$.

Thus as θ varies from -45° to 45°, $\sin\theta+\cos\theta$ increases from 0 to $\sqrt{2}$.

Case II. When $\theta=\frac{\pi}{4}$, then $\sin\theta+\cos\theta=\sqrt{2}$.

As θ increases, $\sin\left(\theta+\frac{\pi}{4}\right)$ decreases (since $\theta+\frac{\pi}{4}>\frac{\pi}{2}$ then and remains positive. When $\theta=\frac{3\pi}{4}$, then

$$\sin\theta+\cos\theta=0.$$

Thus as θ varies from 45° to 135°, $\sin\theta+\cos\theta$ decreases from $\sqrt{2}$ to 0,

Case III. When $\theta=\frac{3\pi}{4}$, $\sin\theta+\cos\theta=0$. As θ increases, $\sin\left(\theta+\frac{\pi}{4}\right)$ now is negative, and it increases in magnitude, therefore $\sin\theta+\cos\theta$ is negative, but increases in magnitude and when $\theta=\frac{5\pi}{4}$ then $\sin\theta+\cos\theta=-\sqrt{2}$.

Thus as θ increases from 135° to 225° $\sin\theta+\cos\theta$ decreases from 0 to $-\sqrt{2}$.

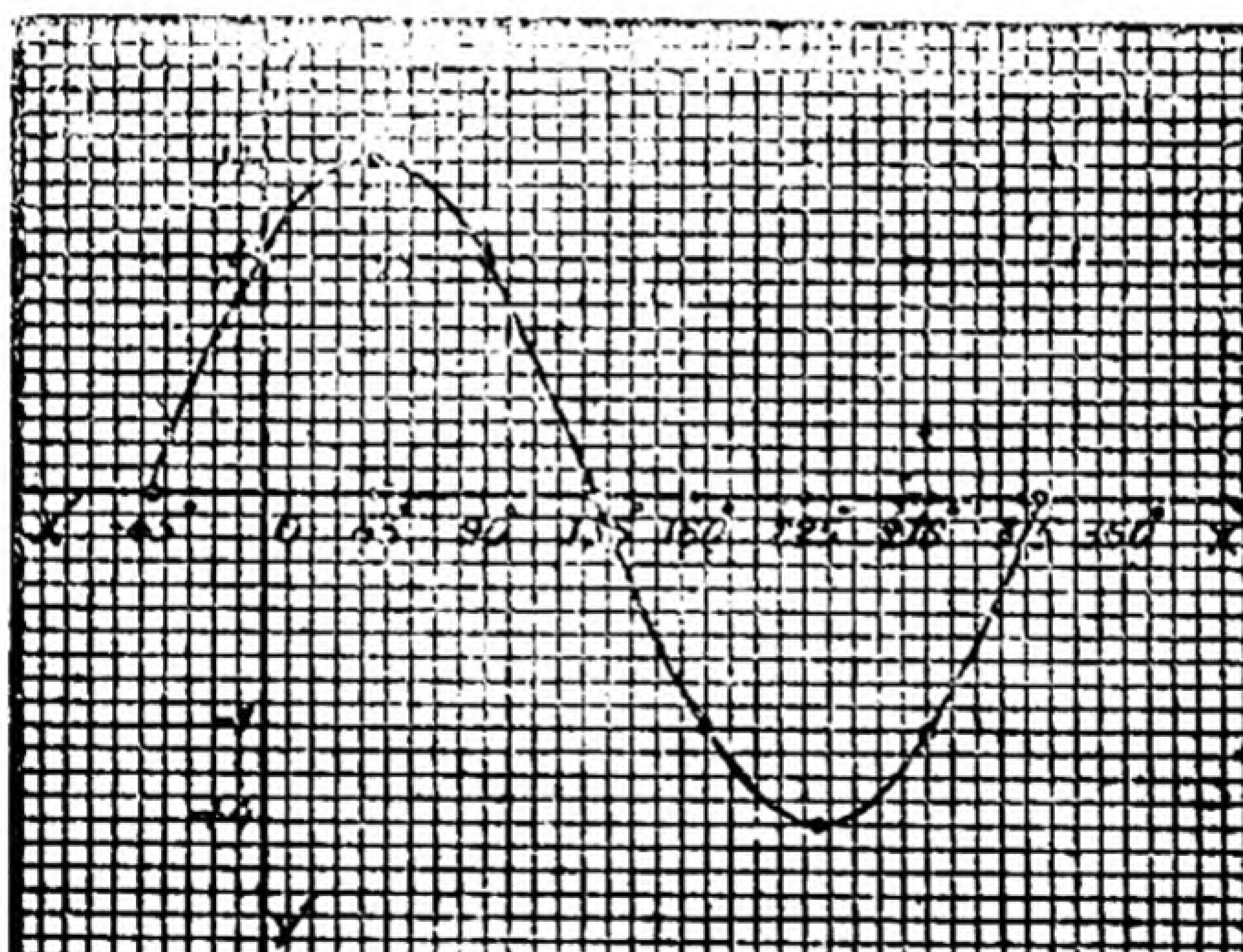
Case IV. When $\theta=\frac{5\pi}{4}$, $\sin\theta+\cos\theta=-\sqrt{2}$.

As θ increases $\sin\left(\theta+\frac{\pi}{4}\right)$ is negative and decreases in magnitude and when $\theta=\frac{7\pi}{4}$ then $\sin\left(\theta+\frac{\pi}{4}\right)=0$.

Thus as θ increases from 225° to 315°, $\sin\theta+\cos\theta$ increases from $-\sqrt{2}$ to 0.

Now if $y=\sin\theta+\cos\theta$, then the following table of values will help to draw the graph as shown in the figure.

$\theta=$	-45°	0	45°	90°	135	180°	225°	270°	315°
$y=$	0	1	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	-1	0



Ex. 4. If θ_1, θ_2 be two values of θ given by $a\cos 2\theta+b\sin 2\theta=c$, then find the values of

(*i*) $\tan\theta_1+\tan\theta_2$, (*ii*) $\tan\theta_1\tan\theta_2$.

Since $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$, $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.

$\therefore$ given equation becomes

$$\frac{a(1-\tan^2\theta)}{1+\tan^2\theta}+\frac{2b\tan\theta}{1+\tan^2\theta}=c$$

or $\tan^2\theta\,(a+c) - 2b\tan\theta + c - a = 0$

$\therefore\ \tan\theta_1 + \tan\theta_2 = \dfrac{2b}{a+c}$

$\tan\theta_1 . \tan\theta_2 = \dfrac{c-a}{c+a}.$

EXERCISE XV

Show that

1. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan(45^\circ - \theta).$
2. $4\cos A\cos(60^\circ - A)\cos(60^\circ + A) = \cos 3A.$
3. $\dfrac{1+\sin 2\theta - \cos 2\theta}{1+\sin 2\theta + \cos 2\theta} = \tan\theta.$
4. Prove that $\dfrac{1+\cos 2\theta - \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \tan\left(\dfrac{\pi}{4} - \theta\right).$
5. $\dfrac{\sin 4\theta}{1+\cos 4\theta} = \dfrac{1-\cos 4\theta}{\sin 4\theta} = \tan 2\theta.$
6. $\dfrac{\sin 2A}{\sin A} - \dfrac{\cos 2A}{\cos A} = \sec A.$
7. $\operatorname{cosec} 2A = \cot A - \cot 2A = \tan A + \cot 2A.$
8. $\sin(2A-B)\cos(2B-A) + \cos(2A-B)\sin(2B-A) = \sin(A+B).$
9. $\dfrac{\cos(A+B)\sin(A-B) + \cos(A-B)\sin(A+B)}{\cos(A-B)\sin(A+B) - \cos(A+B)\sin(A-B)} = \dfrac{\sin 2A}{\sin 2B}.$
10. $\tan 2A\tan 3A\tan 5A = \tan 5A - \tan 3A - \tan 2A.$
11. $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A.$
12. $\tan\dfrac{\pi}{6} + \tan\dfrac{\pi}{12} + \tan\dfrac{\pi}{6}\tan\dfrac{\pi}{12} = 1.$
13. $\cos^2(45^\circ - B) - \sin^2(45^\circ - A) = \sin(A+B)\cos(A-B).$
14. $\cos^4 A - \cos^4 B$
    $= \sin(A+B)\sin(B-A)\{1+\cos(A+B)\cos(A-B)\}.$
15. $\cos^2 2A - \sin^2 A = \cos A\cos 3A.$

16. $\cos^2 A+\cos^2(120°-A)+\cos^2(120°+A)=\frac{3}{2}$.

17. Prove that $\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}=\dfrac{\tan(A+B)}{\cot(A-B)}$.

18. Prove that $\cos(x+y)\cos(x-y)-\sin(x+y)\times\sin(x-y)$ is independent of y.

19. If $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$, show that $\cot D=\cot A\cot B\cot C$.

20. Prove that $\dfrac{\sin 3\theta\cos\theta+\cos 3\theta\sin\theta}{\cos^2 2\theta-\sin^2 2\theta}=\tan 4\theta$.

21. Prove that $\dfrac{\tan A-\tan B}{\cot A+\tan B}=\tan A\tan(A-B)$.

22. Prove that $\cos\theta-\sqrt{3}\sin\theta=2\cos\left(\theta+\frac{\pi}{3}\right)$ and hence find the greatest value of $\cos\theta-\sqrt{3}\sin\theta$.

23. Express $\sqrt{3}\cos\theta+\sin\theta$ in terms of the cosine of single angle.

24. Put $a\cos\theta+b\sin\theta$ in the form $\sqrt{a^2+b^2}\cos(\theta-\alpha)$ and hence find its greatest value. (P. U. 1934, 1942).

25. $\dfrac{1}{\sqrt{2}}\sin\theta=\sin^2\left\{\dfrac{\pi}{8}+\dfrac{\theta}{2}\right\}-\sin^2\left\{\dfrac{\pi}{8}-\dfrac{\theta}{2}\right\}$.

26. Show that (*i*) $\tan 50°=2\tan 10°+\tan 40°$.
(*ii*) $\tan 70°=2\tan 50°+\tan 20°$.

27. Prove that $\dfrac{1-\tan 2A\tan A}{1+\tan 2A\tan A}=\dfrac{\cos 3A}{\cos A}$.

28. Prove that $\dfrac{\tan(45°+A)-\tan(45°-A)}{\tan(45°+A)+\tan(45°-A)}=\sin 2A$.

29. Show that $\tan A\tan(60°+A)\tan(120°+A)=-\tan 3A$.

30. Show that $\cos(A+B)\sin C+\sin(A+B)\cos C$
$=\cos(C+A)\sin B+\sin(C+A)\cos B$
$=\cos(B+C)\sin A+\sin(B+C)\cos A$.

31. Show that $(\sin A\cos B+\cos A\sin B)^2+(\cos A\cos B-\sin A\sin B)^2=1$.

32. If $\frac{a}{b}=\cot A$, prove that

$$\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}=\frac{2\cos A}{\sqrt{\cos 2A}}.$$

33. If $A+B=\frac{\pi}{4}$ prove that $(1+\tan A)(1+\tan B)=2$.

34. Show that $a\cos\theta+b\sin\theta=c$ gives no real value of θ if $c^2>a^2+b^2$.

35. Find numbers a and b which make $a\sin(\theta-30^\circ)+b\sin(\theta+60^\circ)$ identically equal to $2\sin\theta$.

36. Trace the changes in the expression $\sin\theta-\cos\theta$ as θ varies from 0 to 360° and hence draw its graph.

CHAPTER VII

TRANSFORMATION OF PRODUCTS AND SUMS

47. We have already shown that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \quad (1)$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B \quad (2)$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B \quad (3)$$
and
$$\cos(A-B)=\cos A\cos B+\sin A\sin B. \quad (4)$$

Adding the first two equations, we get

$$2\sin A\cos B=\sin(A+B)+\sin(A-B).$$

Subtracting the second from the first equation, we get

$$2\cos A\sin B=\sin(A+B)-\sin(A-B).$$

Adding equations (3) and (4) we get

$$2\cos A\cos B=\cos(A+B)+\cos(A-B).$$

Subtracting equation (3) from equation (4), we get

$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$

V. Imp.

We have thus shown that

Commit to memory

I. $\mathbf{2\sin A\cos B=\sin(A+B)+\sin(A-B)}$.

II. $\mathbf{2\cos A\sin B=\sin(A+B)-\sin(A-B)}$.

III. $\mathbf{2\cos A\cos B=\cos(A+B)+\cos(A-B)}$.

IV. $\mathbf{2\sin A\sin B=\cos(A-B)-\cos(A+B)}$.

Note.—Notice carefully the order of $(A+B)$ and $(A-B)$ in the right-hand side of these formulae, especially in (IV).

These four formulae are useful in changing the products

of two sines, two cosines or a sine and a cosine into a sum or a difference.

Again, put $A+B=P$ and $A-B=Q$,

so that $A=\frac{P+Q}{2}$ and $B=\frac{P-Q}{2}$.

Substituting these values in each of the equations I, II, III and IV, we obtain

$$\text{V.} \quad \sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$

$$\text{VI.} \quad \sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

$$\text{VII.} \quad \cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2},$$

$$\text{and VIII.} \quad \cos P-\cos Q=2\sin\frac{P+Q}{2}\sin\frac{Q-P}{2}.$$

Note.—Notice carefully the order of P and Q in each of these four results, espcially in VIII.

These four formulae are useful in changing the sum or difference of two sines or two cosines into a product.

The eight formulae I.............VIII of this article are of the utmost importance and the student is advised to be thoroughly familiar with them, as no further progress can be made until they haye been thoroughly learnt.

Note 2.—In the above results $\sin\theta+\cos\theta$ is not included but if it is put in the form $\sin\theta+\sin\left(\frac{\pi}{2}-\theta\right)$ then it can be expressed as product.

Note 3.—At the end of the book the student can see the Geometrical method of proving the above result.

Ex. 1. Express $\cos 5\theta-\cos 7\theta$ as a product.

$$\cos 5\theta-\cos 7\theta=2\sin\frac{5\theta+7\theta}{2}\sin\frac{7\theta-5\theta}{2}$$

$$=2\sin 6\theta\sin\theta.$$

Ex. 2. Express $\sin 3\theta\sin 5\theta$ as sum or difference.

$$\sin 3\theta\sin 5\theta=\tfrac{1}{2}\,2\sin 3\theta\sin 5\theta.$$

$$=\tfrac{1}{2}[\cos(5\theta-3\theta)-\cos(5\theta+3\theta)]$$

$$=\tfrac{1}{2}[\cos 2\theta-\cos 8\theta].$$

Ex. 3. Express as a product $\cos 11^\circ + \sin 11^\circ$.

$\cos 11^\circ + \sin 11^\circ = \cos 11^\circ + \cos 79^\circ = 2 \cos 45^\circ \cos 34^\circ$.

Ex. 4. Show that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

$$\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = \cos 20^\circ + (\cos 100^\circ + \cos 140^\circ).$$

$$= \cos 20^\circ + 2 \cos 120^\circ \cos 20^\circ$$
$$= \cos 20^\circ + 2\left(-\tfrac{1}{2}\right) \cos 20^\circ$$
$$= \cos 20^\circ - \cos 20^\circ = 0.$$

Ex. 5. Prove that $\dfrac{\cos A + \cos 3A + \cos 5A + \cos 7A}{\sin A + \sin 3A + \sin 5A + \sin 7A} = \cot 4A$,

The left-hand expression

$$= \frac{(\cos A + \cos 7A) + (\cos 3A + \cos 5A)}{(\sin A + \sin 7A) + (\sin 3A + \sin 5A)}$$

$$= \frac{2 \cos 4A \cos 3A + 2 \cos 4A \cos A}{2 \sin 4A \cos 3A + 2 \sin 4A \cos A}$$

$$= \frac{\cancel{2} \cos 4A \ (\cos 3A + \cos A)}{\cancel{2} \sin 4A \ (\cos 3A + \cos A)} = \cot 4A.$$

Ex. 6. Show that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$.

$$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \tfrac{1}{2} (2 \cos 20^\circ \cos 80^\circ \cos 40^\circ)$$
$$= \tfrac{1}{2}\{(\cos 100^\circ + \cos 60^\circ) \cos 40^\circ\}$$
$$= \frac{1}{2}\left\{\cos 40^\circ \cos(180 - 80)^\circ + \frac{\cos 40^\circ}{2}\right\}$$
$$= \tfrac{1}{4}\{\cos 40^\circ - 2 \cos 40^\circ \cos 80^\circ\}$$
$$= \tfrac{1}{4}\{\cos 40^\circ - (\cos 40^\circ + \cos 120^\circ)\} = \tfrac{1}{8}$$

Or thus :-

$\cos 20^\circ \cos 40^\circ \cos 80^\circ$.

$$= \frac{1}{2 \sin 20^\circ} 2 \sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ$$
$$= \frac{1}{2 \sin 20^\circ} \sin 40^\circ \cos 40^\circ \cos 80^\circ$$
$$= \frac{1}{4 \sin 20^\circ} \sin 80^\circ \cos 80^\circ$$
$$= \frac{1}{8} \cdot \frac{\sin 160^\circ}{\sin 20^\circ} = \frac{\sin 20^\circ}{8 \sin 20^\circ} = \frac{1}{8}.$$

It may be noticed that this method cannot be always employed.

Ex. 7. If θ_1 and θ_2 be two distinct values of θ which satisfy the equation $a \cos \theta + b \sin \theta = c$, prove that

$$\sin(\theta_1+\theta_2) = \frac{2ab}{(a^2+b^2)}.$$

From the given conditions it follows that

$$a \cos \theta_1 + b \sin \theta_1 = c$$
$$a \cos \theta_2 + b \sin \theta_2 = c$$

Subtracting, we get

$$a(\cos\theta_1 - \cos\theta_2) + b(\sin\theta_1 - \sin\theta_2) = 0.$$

or $2a \sin \frac{\theta_1+\theta_2}{2} \sin \frac{\theta_2-\theta_1}{2} + 2b \cos \frac{\theta_1+\theta_2}{2} \sin \frac{\theta_1-\theta_2}{2} = 0,$

or $a \sin \frac{\theta_1+\theta_2}{2} = b \cos \frac{\theta_1+\theta_2}{2}$

so that $\tan \frac{\theta_1+\theta_2}{2} = \frac{b}{a}$.

$$\therefore \sin(\theta_1+\theta_2) = \frac{2\tan\frac{\theta_1+\theta_2}{2}}{1+\tan^2\frac{\theta_1+\theta_2}{2}} = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$$

Ex. 8. Prove the formulae for sin 3A etc, with the help of results of this Chapter.

$$\sin 3A - \sin A = 2\cos 2A \sin A$$
$$= 2(1-2\sin^2 A)\sin A$$
$$= 2\sin A - 4\sin^3 A.$$

By transposition, $\sin 3A = 3\sin A - 4\sin^3 A$.

Similarly $\cos 3A + \cos A = 2\cos 2A \cos A$.

$$= 2(2\cos^2 A - 1)\cos A$$
$$= 4\cos^3 A - 2\cos A.$$

By transposition, $\cos 3A = 4\cos^3 A - 3\cos A$.

Again, $\because \tan 2A = \tan(3A - A)$

$$\therefore \frac{2\tan A}{1-\tan^2 A} = \frac{\tan 3A - \tan A}{1+\tan 3A \tan A}.$$

Cross-multiplication gives

$$(\tan 3A - \tan A)(1-\tan^2 A) = 2\tan A + 2\tan^2 A \tan 3A$$

or $\tan 3A(1-3\tan^2 A) = 3\tan A - \tan^3 A$

$$\therefore \tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}.$$

EXERCISE XVI

Express in the form of a product :

1. $\sin 2\theta+\sin 4\theta$. 2. $\sin 5\theta-\sin 3\theta$.

3. $\cos 3\theta+\sin 3\theta$. 4. $\cos 5\theta-\cos 7\theta$.

5. $\cos 50^\circ-\cos 20^\circ$. 6. $\cos (A+B)+\sin (A-B)$.

Put the following in the form of sum or difference :

7. $2 \sin 2\theta \cos 3\theta$. 8. $2 \cos 5\theta \cos 4\theta$.

9. $2 \sin \theta \sin 3\theta$.

Prove that

10. $\cos 17^\circ-\cos 77^\circ=\cos 43^\circ$.

11. $\dfrac{\cos \theta-\cos 3\theta}{\sin 3\theta-\sin \theta}=\tan \theta$. 12. $\dfrac{\sin 2\theta+\sin 3\theta}{\cos 2\theta-\cos 3\theta}=\cot \dfrac{\theta}{2}$.

13. $\dfrac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cos (A-B) \cot (A+B)$.

14. $\dfrac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan (A-B)$.

15. $\dfrac{\sin A+\sin B}{\sin A \quad \sin B}=\tan \dfrac{A+B}{2} \cot \dfrac{A-B}{2}$.

16. $\dfrac{\cos A+\cos B}{\cos B-\cos A}=\cot \dfrac{A+B}{2} \cot \dfrac{A-B}{2}$.

17. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan \dfrac{A+B}{2}$.

18. $\dfrac{\sin^2 A-\sin^2 B}{\sin A \cos A-\sin B \cos B}=\tan (A+B)$.

19. $\tan \left(\dfrac{A+B}{2}\right)+\tan \dfrac{A-B}{2}=\dfrac{2 \sin A}{\cos A+\cos B}$.

20. $\cos A \cos B=\cos^2 \dfrac{A-B}{2}-\sin^2 \dfrac{A+B}{2}$.

21. $\dfrac{\sin A+2 \sin 3A+\sin 5A}{\sin 3A+2 \sin 5A+\sin 7A}=\dfrac{\sin 3A}{\sin 5A}$.

22. $\dfrac{\sin A+\sin (A+B)+\sin (A+2B)}{\cos A+\cos (A+B)+\cos (A+2B)}=\tan (A+B)$.

23. $\dfrac{\sin\theta+\sin 2\theta+\sin 4\theta+\sin 5\theta}{\cos\theta+\cos 2\theta+\cos 4\theta+\cos 5\theta}=\tan 3\theta.$

24. Show that $\cos 5A+\cos 4A-\cos 3A-\cos 2A$

$$=-4\cos\frac{A}{2}\sin A\ \sin\frac{7}{2}A.$$

25. $\cos(36^\circ-A)\cos(36^\circ+A)$
$+\cos(54^\circ+A)\cos(54^\circ-A)=\cos 2A.$

26. $\cos A\sin(B-C)+\cos B\sin(C-A)$
$+\cos C\sin(A-B)=0.$

27. $\sin(B-C)\sin(A-D)+\sin(C-A)\sin(B-D)$
$+\sin(A-B)\sin(C-D)=0.$

28. $\dfrac{2\sin(A-C)\cos C-\sin(A-2C)}{2\sin(B-C)\cos C-\sin(B-2C)}=\dfrac{\sin A}{\sin B}.$

29. $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}$
$=\cot 6A\cot 5A.$

30. $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}+\dfrac{\sin 2A+\sin 4A}{\cos 2A+\cos 4A}=\dfrac{\sin 5A}{\cos 2A\cos 3A}.$

31. If $A+B+C+D=180^\circ$, then
$\cos 2A-\cos 2B+\cos 2C-\cos 2D$
$=4\sin(A+B)\sin(B+C)\cos(C+A).$

32. Show that

$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$
$-\sin(A+B+C)=4\sin A\sin B\sin C.$

33. Show that $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=\frac{3}{16}.$

34. Show that $\cos 15^\circ+\sin 15^\circ=\dfrac{\sqrt{3}}{\sqrt{2}},$

and $\cos 15^\circ-\sin 15^\circ=\dfrac{1}{\sqrt{2}}.$ [Hint. $\sin 15^\circ=\cos 75^\circ$.]

Hence show that $\cos 15^\circ=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin 15^\circ=\dfrac{\sqrt{3}-1}{2\sqrt{2}}.$

Show that

35. $\cos 10^\circ+\cos 20^\circ+\cos 40^\circ+\cos 50^\circ$
$=\sqrt{3}(\cos 10^\circ+\cos 20^\circ).$

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36. $\sin 10^\circ+\sin 20^\circ+\sin 40^\circ+\sin 50^\circ=\sin 70^\circ+\sin 80^\circ$.

37. $\dfrac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ}=\tan 56^\circ$.

38. $2\sin\dfrac{\pi}{9}\left(\cos\dfrac{2\pi}{9}+\cos\dfrac{4\pi}{9}+\cos\dfrac{6\pi}{9}+\cos\dfrac{8\pi}{9}\right)$

$$=-\sin\frac{8\pi}{9}.$$

39. $\sin A\sin(A+2B)-\sin B\sin(B+2A)$
$$=\sin(A-B)\sin(A+B).$$

40. If $\sin A\sin C=\sin(B+A)\sin(B+C)$, prove that either $A+B+C$ or B is a multiple of π.

41. If θ_1 and θ_2 be two different and distinct values of θ which satisfy $a\cos\theta+b\sin\theta=c$, show that

$$\tan(\theta_1+\theta_2)=\frac{2ab}{b^2-a^2}.$$

42. Find the greatest value of $\sin\theta\sin(\alpha-\theta)$, α being a given acute angle.

Important Formulae on Chapter VI and VII.

I. $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$\sin 2A=2\sin A\cos A$

$$=\frac{2\tan A}{1+\tan^2 A}.$$

$\cos 2A=1-2\sin^2 A=2\cos^2 A-1$

$$=\frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

$\sin 3A=3\sin A-4\sin^3 A$

$\cos 3A=4\cos^3 A-3\cos A$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

$$\sin (A-B) = \sin A \cos B - \cos A \sin B.$$

$$(\cos A - B) = \cos A \cos B + \sin A \sin B.$$

$$\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

$$\sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B.$$

$$\cos (A+B) \cos (A-B) = \cos^2 A - \sin^2 B$$
$$= \cos^2 B - \sin^2 A.$$

$$\tan (A+B+C)$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$$

$$\sin A \cos B = \tfrac{1}{2} [\sin (A+B) + \sin (A-B)].$$

$$\cos A \sin B = \tfrac{1}{2} [\sin (A+B) - \sin (A-B)].$$

$$\cos A \cos B = \tfrac{1}{2} [\cos (A+B) + \cos (A-B)].$$

$$\sin A \sin B = \tfrac{1}{2} [\cos (A-B) - \cos (A+B)].$$

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} . \cos \frac{P-Q}{2}.$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} . \sin \frac{P-Q}{2}.$$

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} . \cos \frac{P-Q}{2}.$$

$$\cos P - \cos Q = 2 \sin \frac{Q+P}{2} . \sin \frac{Q-P}{2}.$$

REVISION QUESTIONS V

1. Show that $\sin (60^\circ + \theta) - \sin (60^\circ - \theta) - \sin \theta = 0$.
2. Prove that $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$.
3. $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$.
4. $\dfrac{\cos 13^\circ + \sin 13^\circ}{\cos 13^\circ - \sin 13^\circ} = \tan 58^\circ$.
5. Prove that $\dfrac{1 - \cos A + \cos B - \cos (A+B)}{1 + \cos A - \cos B - \cos (A+B)} = \dfrac{\tan \frac{A}{2}}{\tan \frac{B}{2}}$.
6. Prove that $\dfrac{\sin (A-B) + \sin A + \sin (A+B)}{\sin (C-B) + \sin C + \sin (C+B)} = \dfrac{\sin A}{\sin C}$.

7. Show that if an angle α be divided into two parts such that the ratio of tangents of the parts is n, the difference β between the parts is given by $\sin\beta = \frac{n-1}{n+1}\sin\alpha$.

8. If $\frac{\sin(A+B)}{\cos(A-B)} = \frac{1-\lambda}{1+\lambda}$, prove that

$$\tan\left(\frac{\pi}{4} - B\right) = \lambda \cot\left(\frac{\pi}{4} - A\right).$$

9. If $b\sin\beta = a\sin(2\alpha+\beta)$, show that.

$$(b+a)\cot(\alpha+\beta) = (b-a)\cot\alpha.$$

10. If $\sin\theta = a\sin(\theta+2\alpha)$, then $\tan(\theta+\alpha) = \frac{1+a}{1-a}\tan\alpha$.

11. If $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$, find $\tan\frac{\theta+\phi}{2}$.

12. Prove that $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$.

13. If $\frac{a}{b} = \frac{\cos A}{\cos B}$, prove that

$$a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}.$$

14. If $\frac{\sin(\beta+\theta)}{\sin(\beta-\theta)} = \frac{\cos(30^\circ-\phi)}{\cos(30^\circ+\phi)}$, show that

$$\tan\theta = \frac{1}{\sqrt{3}}\tan\beta\tan\phi.$$

15. If $\tan^2\theta = \tan(\theta-\alpha)\tan(\theta-\beta)$, prove that,

$$\tan 2\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

16. If $x\cos\theta = y\cos\left(\theta + 2\frac{\pi}{3}\right) = z\cos\left(\theta + 4\frac{\pi}{3}\right)$.

prove that $xy + yz + zx = 0$.

48. Some General Theorems :—

(1) Sum up the series

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin(\alpha+\overline{n-1}\beta)$$

Let $S=\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots+\sin(\alpha+\overline{n-1}\beta)$,

then $2\sin\frac{\beta}{2}S=2\sin\frac{\beta}{2}\sin\alpha+2\sin\frac{\beta}{2}\sin(\alpha+\beta)$

$+2\sin\frac{\beta}{2}\sin(\alpha+2\beta)+\ldots\ldots+2\sin\frac{\beta}{\alpha}\sin(\alpha+n-1)\beta$

But $2\sin\frac{\beta}{2}\sin\alpha=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)$

$2\sin\frac{\beta}{2}\sin(\alpha+\beta)=\text{ccs}\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right)$

$\sin\frac{\beta}{2}\sin(\alpha+2\beta)=\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right)$

....

and finally

$$2\sin\frac{\beta}{2}\sin(\alpha+\overline{n-1}\beta)=\cos\left(\alpha+\frac{2n-3}{2}\beta\right)$$
$$-\cos\left(\alpha+\frac{2n-}{2}\beta\right).$$

Adding all these we get

$$2\sin\frac{\beta}{2}S=\left[\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)\right]$$

$$=2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\frac{\sin n\beta}{2}.$$

$$\therefore\quad S=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

(2) Sum up the series

$\cos\alpha+\cos(\alpha+\beta)+\ldots\ldots\ldots+\cos(\alpha+\overline{n-1}\beta)$.

Let S be the given series. Multiplying by $2\sin\frac{\beta}{2}$ and proceeding as above, the result is easily obtained. The sum required is $\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}$.

EXERCISE XVII

Sum up the series :—

1. $\cos\alpha+\cos 3\alpha+\cos 5\beta+\ldots\ldots$ to n terms.
2. $\sin 2\alpha+\sin 4\alpha+\sin 6\alpha+\ldots\ldots$ to n terms
3. $\cos\alpha\sin\alpha+\cos 2\alpha\sin 2\alpha+\cos 3\alpha\sin 3\alpha+\ldots\ldots$ to n terms.
4. $\sin\alpha\cos 2\alpha+\sin 2\alpha\cos 3\alpha+\sin 3\alpha\cos 4\alpha+\ldots\ldots$ to n terms.

[**Hint.** Put each term as the difference of two terms.]

5. $\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots\ldots$ to n terms.

CHAPTER VIII
SUB-MULTIPLE ANGLES

49. To express the trigonometrical functions of an angle in terms of the cosine of double the angle.

We have $\cos 2A=1-2\sin^2A=2\cos^2A-1$.

Hence $2\sin^2A=1-\cos 2A$ and $2\cos^2A=1+\cos 2A$.

$$\therefore \sin A=\pm\sqrt{\frac{1-\cos 2A}{2}} \text{ and } \cos A=\pm\sqrt{\frac{1+\cos 2A}{2}}.$$

$$\therefore \tan A=\pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}.$$

Taking the reciprocal's we obtian the other functions.

When only $\cos 2A$ is given and nothing more is said about A or 2A, the ambiguity of signs in the foregoing results cannot be removed. But when in addition to $\cos 2A$, A is given or if we know in which quadrant A lies, the ambiguity can be easily removed.

Ex. 1. Find $\sin 22^\circ 30'$, $\cos 22^\circ 30'$ and $\tan 22^\circ 30'$,

Put $A=22^\circ 30'$; $\therefore 2A=45^\circ$.

$$\therefore \sin 22^\circ 30'=\pm\sqrt{\frac{1-\cos 45^\circ}{2}}=\pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

$$=\pm\frac{1}{2}\sqrt{2-\sqrt{2}}.$$

Similarly $\cos 22^\circ 30'=\pm\frac{1}{2}\sqrt{2+\sqrt{2}}$.

Now $\because$ A lies in first quadrant, therefore $\sin A$ and $\cos A$ are positive. Hence rejecting the negative sign,

we have

$$\sin\ 22°\,30' = \frac{\sqrt{2-\sqrt{2}}}{2} \text{ and } \cos 22°\,30' = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\therefore\ \tan 22°\,30' = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}.$$

Ex. 2. Find sin 15° and cos 15°.
Put $A=15°$, and $\therefore\ 2A=30°$.

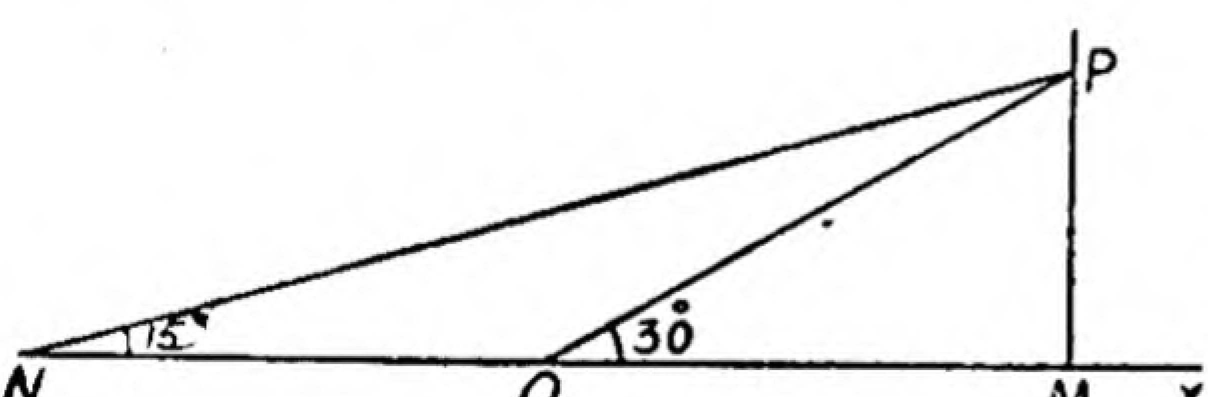
$$\therefore\ \sin\ 15° = \pm\sqrt{\frac{1-\cos\ 30°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \pm\frac{\sqrt{3}-1}{2\sqrt{2}} \text{ and similarly } \cos\ 15° = \pm\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Now $\because$ A lies in first quadrant, therefore sin A is positive and cos A is also positive. Hence rejecting the lower sign in the first as well as in the second result we have

$$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} \text{ and } \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

It may be noticed that these results agree with what we had obtained already. However, they can also be proved geometrically thus :



Let angle XOP be equal to 30°. Take OP equal to two units and draw PM perpendicular to OX,

Produce MO to N such that ON=OP. Join PN.

Then $MP=1$ and $OM=\sqrt{3}$; also $ON=OP=2$.

Therefore $NP=\sqrt{MP^2+NM^2}=\sqrt{1+(2+\sqrt{3})^2}$

$$=\sqrt{8+4\sqrt{3}}=\sqrt{2}(1+\sqrt{3})$$

$$\text{Hence } \sin 15° = \frac{MP}{NP} = \frac{1}{\sqrt{2}(1+\sqrt{3})} = \frac{(\sqrt{3}-1)}{2\sqrt{2}}$$

$$\cos 15° = \frac{NM}{NP} = \frac{2+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})} = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

$$\tan 15° = \frac{MP}{NM} = \frac{1}{2+\sqrt{3}} = \frac{2}{4+2\sqrt{3}}$$

$$=\frac{(\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)^2}=\frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

Note 1. Since $\angle MPN=75°$, the same figure can be used to find the circular functions of 75°.

Note 2.—The method may be extended to angles of $7\frac{1}{2}°$, $3\frac{3}{4}°$ and the like.

Note 3. A similar construction (starting with $\angle MOP=45°$) gives the circular functions of $22\frac{1}{2}°$, $11\frac{1}{4}$ and the like.

Note.—From the above we derive a very interesting result :

$$2\cos\frac{A}{2}=2\sqrt{\tfrac{1}{2}(1+\cos A)}$$

$$=\sqrt{4\times\tfrac{1}{2}(1+\cos A)}=\sqrt{(2+2\cos A)}.$$

Changing A to $\frac{A}{2}$, we get $2\cos\frac{A}{2^2}$

$$=\sqrt{\left(2+2\cos\frac{A}{2}\right)}=\sqrt{\{2+\sqrt{(2+2\cos A)}\}}$$

Similarly $2\cos\frac{A}{2^3}=\sqrt{\left(2+2\cos\frac{A}{2^2}\right)}$

$$=\sqrt{[2+\sqrt{\{2+\sqrt{(2+2\cos A)}\}}]}.$$

In the above statement A is supposed to be less than two right angles, so that all the radicals are taken with the sign *plus* before them.

Also it is clear that when the denominator of A is 2^n the radical sign will appear n times on the right-hand side.

Thus

$$2\cos\frac{A}{2^n}=\sqrt{(2+\sqrt{(2+\sqrt{(2+\ldots+\sqrt{(2+2\cos A)})\ldots})}}.$$

Squaring this and adding to the square of $2\sin\frac{A}{2^n}$ we get

$$4=4\sin^2\frac{A}{2^n}+2+\sqrt{(2+\ldots\sqrt{(2+2\cos A)})\ldots)}$$

$$\therefore\ 4\sin^2\frac{A}{2^n}=2+\sqrt{(2+\ldots\sqrt{((2+2\cos A))\ldots)}}$$

$$\therefore\ 2\sin\frac{A}{2^n}=\sqrt{2+\sqrt{(2+\sqrt{(2+\ldots+\sqrt{(2+2\cos A)})\ldots)}}}$$

the radical sign appearing n times as before.

Thus $2\cos 22°\,30'=2\cos\frac{180°}{2^3}$

$=\sqrt{(2+\sqrt{(2+\sqrt{(2+2\cos 180^\circ)}))}}=\sqrt{(2+\sqrt{2})}$

$\therefore \cos 22^\circ 30'=\frac{1}{2}\sqrt{(2+\sqrt{2})}$ as before.

50. To express the trigonometrical functions of an angle in terms of the sine of double the angle.

$$\sin^2 A+\cos^2 A=1$$

$$\text{and } 2\sin A\cos A=\sin 2A.$$

Adding and taking the square root, we get,

$\sin A+\cos A=\pm\sqrt{(1+\sin 2A)}$. ...(*i*)

Substracting and taking the square root, we get

$$\sin A-\cos A=\pm\sqrt{(1-\sin 2A)} \quad \ldots(ii)$$

From (*i*) and (*ii*) by adding and subtracting, we get

$\sin A=\frac{1}{2}(\pm\sqrt{1+\sin 2A}\pm\sqrt{1-\sin 2A})$

$\cos A=\frac{1}{2}(\mp\sqrt{1+\sin 2A}\mp\sqrt{1-\sin 2A})$.

Dividing sin A by cos A, we obtain tan A in terms of sin 2 A. By taking reciprocals, we get the remaining functions.

When only sin 2A is given and nothing more is said about A the ambiguity of signs in the foregoing results cannot be removed. But when in addition to sin 2A, **A** is given, the ambiguity can be easily removed.

Ex. 1. Find $\sin 9^\circ$ and $\cos 9^\circ$, given that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$.

Put $A=9^\circ$ in (*i*) and (*ii*) above.

$\therefore \sin 9^\circ+\cos 9^\circ=\pm\sqrt{1+\sin 18^\circ}$ ...(*i*)

and $\sin 9^\circ-\cos 9^\circ=\pm\sqrt{1-\sin 18^\circ}$ ...(*ii*)

Now since 9° lies in the first quadrant, therefore both sin 9° and cos 9° are positive Hence we reject the lower sign in (*i*). Also because $\cos 9^\circ>\sin 9^\circ$, therefore we reject the upper sign in (*ii*). Thus ambiguity of signs is removed. Hence

$$\sin 9^\circ+\cos 9^\circ=+\sqrt{1+\sin 18^\circ}=\frac{\sqrt{3+\sqrt{5}}}{2}$$

$$\text{and } \sin 9^\circ-\cos 9^\circ=-\sqrt{1-\sin 18^\circ}=-\frac{\sqrt{5-\sqrt{5}}}{2}$$

Adding and subtracting these,

$$\sin 9^\circ=\frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}.$$

$$\cos 9^\circ = \frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}.$$

Ex. 2. Given that $\sin A = \frac{24}{25}$ and that **A** lies between 90° and 120°, find $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$.

Here $\frac{A}{2}$ lies between 45° and 60°; therefore both $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ are positive. Hence for this case the upper sign must be retained in (*i*). Also $\cos \frac{A}{2} < \sin \frac{A}{2}$, therefore the upper sign must be retained in (*ii*).

$$\therefore \quad \sin \frac{A}{2} + \cos \frac{A}{2} = +\sqrt{1+\sin A} = \frac{7}{5},$$

$$\text{and } \sin \frac{A}{2} - \cos \frac{A}{2} = +\sqrt{1-\sin A} = \frac{1}{5}$$

$$\therefore \quad \sin \frac{A}{2} = \frac{4}{5} \text{ and } \cos \frac{A}{2} = \frac{3}{5}.$$

Note:—For the convenience of the student we give here the signs which the expression $\sin A + \cos A$ and $\sin A - \cos A$ assume as A changes from 0 to 360°; the results can be very easily verified :—

I. If A lies between $\frac{-\pi}{4}$ and $\frac{\pi}{4}$,

then $\sin A + \cos A$ is positive and $\sin A - \cos A$ is negative.

II. If A lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$,

then $\sin A + \cos A$ is positive and $\sin A - \cos A$ is also positive

III. If A lies between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$,

then $\sin A + \cos A$ is negative and $\sin A - \cos A$ is positive.

IV. If A lies between $\frac{5\pi}{4}$ and $\frac{-\pi}{4}$,

then $\sin A + \cos A$ is negative and $\sin A - \cos A$ is positive.

51. To express the trigonometrical ratios of an angle in terms of the tangent of double the angle.

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

By cross-multiplication and transposition we get

$\tan 2A \tan^2 A + 2\tan A - \tan 2A = 0,$

$$\tan A = \frac{-2 \pm \sqrt{4+4\tan^2 2A}}{2\tan 2A} = \frac{-1 \pm \sqrt{1+\tan^2 2A}}{\tan 2A}.$$

Thus tan A is found in terms of tan 2A and hence other functions of A can be determined.

Note 1.—From the foregoing quadratic equation we see that one value of tan A is the reciprocal of the other with its sign changed.

Note 2.—When nothing is known about A, sin A, cos A have in general four values each.

Ex. Given $\tan 2A = -\frac{120}{119}$, find sin A and cos A.

$$-\frac{120}{119} = \tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$\therefore \quad 120 - 120\tan^2 A = -119 \times 2\tan A$

or $60\tan^2 A - 119\tan A - 60 = 0$

or $(5\tan A - 12)(12\tan A - 5) = 0$

$$\therefore \quad \tan A = \frac{12}{5} \text{ or } -\frac{5}{12}.$$

Whence $\sin A = \pm\frac{12}{13}, \ \pm\frac{5}{13}$, and $\cos A = \pm\frac{5}{13}, \ \pm\frac{12}{13}$.

52. To find the circular functions of 18° and 72°.

Let $18° = \theta$, so that $5\theta = 90°$ or $2\theta = 90° - 3\theta$

$\therefore \quad \sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta$

or $2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

Divide both sides by $\cos\theta$ (which is not zero) ;

$$\begin{aligned}\therefore \quad 2\sin\theta &= 4\cos^2\theta - 3\\ &= 4(1-\sin^2\theta) - 3\\ &= 1 - 4\sin^2\theta.\end{aligned}$$

Transposition gives $4\sin^2\theta + 2\sin\theta - 1 = 0$

$$\therefore \quad \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}.$$

Now 18° is an acute angle and hence its sine is positive. Hence rejecting the negative value, we have

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$

Again, $\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)}$

$= \sqrt{1 - \frac{6-2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}.$

The remaining circular functions can now be found.
Since 18° and 72° are complimentary angles, therefore,

$\sin 72° = \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4},$

and $\cos 72° = \sin 18° = \frac{\sqrt{5}-1}{4}.$

Sine and cosine being known, all other trigonometrical ratios of 18° and 72° can be calculated.

53. To find the circular functions of 36° and 54°

Let $36° = \theta$, so that
$5\theta = 180°$ or $2\theta = 180° - 3\theta$.
$\therefore \sin 2\theta = \sin(180° - 3\theta) = \sin 3\theta$.
or $2 \sin\theta \cos\theta = 3 \sin\theta - 4 \sin^3\theta$.
Divide both sides by $\sin\theta$ (which is not zero) ;
$\therefore 2\cos\theta = 3 - 4(1 - \cos^2\theta)$
$= -1 + 4\cos^2\theta$
$4\cos^2\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \frac{2 \pm \sqrt{4+16}}{8} = \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{1}.$

Now 36° is an acute angle and therefore its cosine is positive. Hence rejecting the negative value, we have

$\cos 36° = \frac{\sqrt{5}+1}{4}.$

$\therefore \sin 36° = \sqrt{1 - \cos^2 36°} = \sqrt{1 - \left(\frac{\sqrt{5}+1}{4}\right)^2}$

$= \sqrt{1 - \frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

The remaining circular functions can now be found.
Since 36° and 54° are complimentary angles, therefore

$\sin 54° = \cos 36° = \frac{\sqrt{5}+1}{4},$

and $\cos 54° = \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}$

The value of cos 36° can also be deduced from that of sin 18° thus :—

$$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}.$$

Thus sine and cosine being known, all other trigonometrical functions of 54° and 36° can be calculated.

EXERCISE XVIII

1. In the formula $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, find the sign to be taken when A is (*i*) 120°. (*ii*) 240°, (*iii*) 400°.

2. In the formula $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, find the sign to be taken when A is (*i*) 48°, (*ii*) 302°, (*iii*) 400°, (*iv*) 560°.

3. In the formula $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$, find the sign to be taken when A is (*i*) 26°, (*ii*) 200°, (*iii*) 400°,

4. Find $\sin 157\frac{1}{2}^\circ$ and $\cos 157\frac{1}{2}^\circ$.

5. Find $\sin 292\frac{1}{2}^\circ$ and $\cos 292\frac{1}{2}^\circ$. $\to 292\frac{1}{2}^{\circ}$

6. Find $\tan\frac{A}{2}$, $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ if $\tan A = \frac{21}{20}$, and $\frac{A}{2}$ lie in the first quadrant.

7. In the formula $\cos\frac{A}{2} + \sin\frac{A}{2} = \pm\sqrt{1+\sin A}$, find the sign to be taken when A is (*i*) 80°, (*ii*) 280°, (*iii*) 380°.

8. In the formula $\cos\frac{A}{2} - \sin\frac{A}{2} = \pm\sqrt{1-\sin A}$, find the sign to be taken when $\frac{A}{2}$ is (*i*) 35°. (*ii*) 50° (*iii*) 100°.

9. **Given that $\sin 30° = \frac{1}{2}$, deduce the values of $\sin 15°$ and $\cos 15°$.**

10. Given that $\sin 60° = \frac{\sqrt{3}}{2}$ deduce the known values of $\sin 30°$ and $\cos 30°$.

11. If $A = 340°$, prove that

$$2 \sin \frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A};$$

and $$2 \cos \frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

12. If $A = 580°$ prove that

$$2 \sin \frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

13. If θ is an acute angle and $\sin \theta = \frac{2ab}{a^2+b^2}$, find $\tan \frac{\theta}{2}$.

TRIGONOMETRICAL IDENTITIES

54. When two or more angles are connected by some relation, we can find a relation existing among their circular functions.

The method of discovering such a relation is best illustrated by examples.

The student is advised to notice carefully the several steps.

R **Ex. 1.** $A+B+C=\pi$, show that
$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

We have $\sin 2A + \sin 2B + \sin 2C$

$$= 2 \sin (A+B) \cos (A-B) + 2 \sin C \cos C.$$
$$= 2 \sin (\pi - C) \cos (A-B) + 2 \sin C \cos [\pi - (A+B)]$$

[Note this step.]

$$= 2 \sin C \cos (A-B) - 2 \sin C \cos (A+B)$$
$$= 2 \sin C [\cos (A-B) - \cos (A+B)]$$
$$= 2 \sin C (2 \sin A \sin B)$$
$$= 4 \sin A \sin B \sin C.$$

R **Ex. 2.** If $A+B+C=\pi$, show that

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

We have

$$\cos A+\cos B+\cos C=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$$

$$=1+2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}$$

$$-2\sin\frac{C}{2}\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)$$

$$=1+2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{A+B}{2}$$

$$=1+2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}$$

$$=1+2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. If $A+B+C=\pi$, show that

$$\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C.$$

We have $\sin^2 A+\sin^2 B-\sin^2 C$

$=\sin^2 A+\sin(B+C)\sin(B-C)$

$=\sin A\sin[\pi-(B+C)]+\sin(\pi-A)\sin(B-C)$

$=\sin A\sin(B+C)+\sin A\sin(B-C)$

$=\sin A[\sin(B+C)+\sin(B-C)]$

$=2\sin A\sin B\cos C.$

Ex. 4. If $A+B+C=\pi$, show that

$$\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C.$$

We have $\cos^2 A+\cos^2 B-\cos^2 C$

$=\frac{1}{2}[2\cos^2 A+2\cos^2 B-2\cos^2 C]$

$=\frac{1}{2}[1+\cos 2A+1+\cos 2B-2\cos^2 C]$

$=\frac{1}{2}[2+\cos 2A+\cos 2B-2\cos^2 C]$

$=\frac{1}{2}[2+2\cos(A+B)\cos(A-B)-2\cos^2 C]$

$=\frac{1}{2}[2+2\cos(\pi-C)\cos(A-B)-2\cos C$
$\times\cos(\pi-A-B)]$

$=\frac{1}{2}[2-2\cos C\cos(A-B)+2\cos C\cos(A+B)]$

$=\frac{1}{2}[2-2\cos C\{\cos(A-B)-\cos(A+B)\}]$

$=\frac{1}{2}[2-4\sin A\sin B\cos C]$

$=1-2\sin A\sin B\cos C.$

Anothsr Method :

$\cos^2 A+\cos^2 B-\cos^2 C=\cos^2 A+1-\sin^2 B-\cos^2 C$

$=1+(\cos^2 A-\sin^2 B)-\cos^2 C$

$=1+\cos(A+B)\cos(A-B)-\cos^2 C$
$=1+\cos(180°-C)\cos(A-B)-\cos^2 C$
$=1-\cos C\cos(A-B)-\cos^2 C$
$=1-\cos C[\cos(A-B)+\cos C]$
$=1-\cos C[\cos(A-B)+\cos\{180°-(A+B)\}]$
$=1-\cos C[\cos(A-B)-\cos(A+B)]$
$=1-\cos C\times 2\sin A\sin B$
$=1-2\sin A\sin B\cos C.$

Note.—Any relation which exists on the supposition that $A+B+C=\pi$ must also hold when A, B, C are changed into $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$ or $\pi-2A$, $\pi-2B$ and $\pi-2C$ respectively; for on the same supposition, the sum of these three angles is also equal to π.

Thus from corollary to Article 45 we get

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

Ex 5. In a triangle ABC if $\sin 2B+\sin 2C=\sin 2A$, show that either $B=90°$ or $C=90°$.

Here $\sin 2B+\sin 2C=\sin 2A$
or $2\sin(B+C)\cos(B-C)=2\sin A\cos A$
or $\sin A\cos(B-C)=\sin A\cos A$
or $\sin A[\cos(B-C)+\cos(B+C)]=0$
or $2\sin A\cos B\cos C=0.$

Now A can neither be 0 nor 180°, therefore either $\cos B=0$ or $\cos C=0$, $\therefore$ B or $C=90°$.

Ex. 6. If $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, find all the possible relations between α, β, γ.

We will transpose 1 to the left and deal with the resulting equation. Thus

$$\cos\alpha+\cos\beta+\cos\gamma-1=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+1-2\sin^2\frac{\gamma}{2}-1$$
$$=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\sin^2\frac{\gamma}{2}.$$

Now this expression is to be equal to $4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

Therefore $\sin\frac{\gamma}{2}$ must be a factor of this expression

and hence $\sin\frac{\gamma}{2}$ must be equal to either $\cos\frac{\alpha+\beta}{2}$ or $\cos\frac{\alpha-\beta}{2}$. Let, if possible, $\cos\frac{\alpha-\beta}{2}=\sin\frac{\gamma}{2}$.

Hence $2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\sin^2\frac{\gamma}{2}$

$$=2\sin\frac{\gamma}{2}\left\{\cos\frac{\alpha+\beta}{2}-\sin\frac{\gamma}{2}\right\}$$

$$=2\sin\frac{\gamma}{2}\left\{\cos\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right\}$$

$$=-4\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

which shows that $\cos\frac{\alpha-\beta}{2}=\sin\frac{\gamma}{2}$ does not fit in.

Therefore we take now $\cos\frac{\alpha+\beta}{2}=\sin\frac{\gamma}{2}$ (*a*) and now the expression $=2\sin\frac{\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2}-\sin\frac{\gamma}{2}\right\}$

$$=2\sin\frac{\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right\}$$

$$=4\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Therefore from (*a*) we have $\cos\frac{\alpha+\beta}{2}=\sin\frac{\gamma}{2}=\cos\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)$

$$\therefore\quad \frac{\alpha+\beta}{2}=2n\pi\pm\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)$$

or $\alpha+\beta=4n\pi\pm(\pi-\gamma)$,

which gives all the possible relations between α, β and γ.

EXERCISE XIX

If $A+B+C=180°$ prove that

1. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.
2. $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$.
3. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$.
4. $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.
5. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}$

6. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

7. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.
(P. U. 1945)

8. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

9. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

10. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{2}\sin\frac{\pi-B}{2}\times$
$\sin\frac{\pi-C}{4}$.

11. $\cos^2A+\cos^2B+\cos^2C=1-2\cos A\cos B\cos C$.

12. $\cos^2A+\cos^2B-\cos^2C=1-2\sin A\sin B\cos C$.

13. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

14. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

15. $\sin 3A+\sin 3B+\sin 3C=-4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}$.

16. $\cos 4A+\cos 4B+\cos 4C=4\cos 2A\cos 2B\cos 2C-1$.

17. $\sin^2 2A+\sin^2 2B+\sin^2 2C=2-2\cos 2A\cos 2B\cos 2C$,

18. $\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\times$
$\cos\frac{\pi-C}{4}$.

19. $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$
$=4\sin A\sin B\sin C$.

20. $\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}$
$+\cos\frac{C}{2}\cos\frac{A-B}{2}=\sin A+\sin B+\sin C$.

21. $\tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C$.
Hence show that if $x+y+z=xyz$, then

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}.$$

22. $\tan A \cot B \cot C + \tan B \cot C \cot A + \tan C \cot A \cot B = \tan A + \tan B + \tan C - 2(\cot A + \cot B + \cot C)$.

23. $\dfrac{\cot A+\cot B}{\tan A+\tan B}+\dfrac{\cot B+\cot C}{\tan B+\tan C}+\dfrac{\cot C+\cot A}{\tan C+\tan A}=1.$

24. If $A+B+C=180°$ show that
$\sin^2(A/2+B)+\sin^2(B/2+C)+\sin^2(C/2+A)$
$=1+2\sin(A/2+B)\sin(B/2+C)\sin(C/2+A)$ (B.U.)

25. If $A+B+C+D=2\pi$, prove that
$\sin A-\sin B+\sin C-\sin D$

$$=-4\cos\frac{A+B}{2}\sin\frac{A+C}{2}\cos\frac{A+D}{2}.$$

26. If $A+B+C+D=2\pi$, show that

$$\cos A+\cos B+\cos C+\cos D=4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\times\cos\frac{A+D}{2}.$$

27. If $\alpha+\beta=\gamma$, show that
$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma$.

28. Prove that if $A\pm B\pm C$ is zero or a multiple of 2π,
$\cos^2A+\cos^2B+\cos^2C-2\cos A\cos B\cos C=1$.

29. Show that the value of the expression $\sin B\sin C\times\cos A+\sin^2A$ remains unchanged if any two of the letters A, B, C are interchanged, provided that A, B C are the angles of a triangle

If A, B, C are any angles and $2S=A+B+C$, prove that

30. $4\sin S\sin(S-A)\sin(S-B)\sin(S-C)=$
$1-\cos^2A-\cos^2B-\cos^2C+2\cos A\cos B\cos C$.

31. $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S$

$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Formulae of Chapter VIII

1. $\sin A=\pm\sqrt{\dfrac{1-\cos 2A}{2}}$; 2. $\cos A=\pm\sqrt{\dfrac{1+\cos 2A}{2}}$

3. $\tan A=\pm\sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$; 4. $\sin 18°=\dfrac{\sqrt5-1}{4}=\cos 72°$.

5. $\cos 18°=\dfrac{\sqrt{10+2\sqrt5}}{4}=\sin 72°$.

6. $\cos 36^\circ = \sin 54^\circ = \dfrac{\sqrt{5}+1}{4}.$

7. $\cos 54^\circ = \sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}.$

MISCELLANEOUS EXERCISES II

1. If A and B are acute angles given by $\cos A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$. calculate the value of $\tan (A+B)$.

2. If $\cos A = \frac{1}{7}$ and $\cos B = \frac{13}{14}$ (A, B being acute) prove that $A - B = 60^\circ$.

3. Verify that the formulæ for $\cos (60^\circ - 45^\circ)$ and $\cos (45^\circ - 30^\circ)$ lead to the same value of $\cos 15^\circ$.

4. Find $\sin 90^\circ$ and $\cos 90^\circ$ with the help of he formulæ for $\sin (A+B)$ and $\cos (A+B)$.

5. Draw the graph of $\sin x$, x varying from 0° to 360°.

6. Draw the graph of $-\cos x$, x varying from 0° to 360°.

7. Show that $\dfrac{\sin 4A}{\sin A} = 2 (\cos 3A + \cos A)$.

8. Show that $\cos^2 A + \cos^2 (120^\circ - A) + \cos^2 (120^\circ + A) = \frac{3}{2}$.

9. Prove that

$$\tan \theta + \tan (60^\circ + \theta) + \tan (120^\circ + \theta) = 3 \tan 3\theta.$$

10. If $\sin \theta = \dfrac{m^2 - n^2}{m^2 + n^2}$, find $\tan \dfrac{\theta}{2}$.

11. Show that (*i*) $\dfrac{\cot A - \tan A}{\cot A + \tan A} = \cos A$;

(*ii*) $\tan A + \cot A = 2 \operatorname{cosec} 2A$.

12. Show that $\sin 2A \sin A = 2 \cos A - \cos^3 A$.

13. If $\cot \theta = \dfrac{a}{b}$, find $\sin 2\theta$.

14. Prove the identities

(*i*) $\tan (45^\circ + \theta) + \tan (45^\circ - \theta) = 2 \sec 2\theta$.

(*ii*) $\operatorname{cosec} 2A + \cot 4A = \cot A - \operatorname{cosec} 4A$.

15. Prove that $\dfrac{1 + \sin 2\theta}{1 - \sin 2\theta} = \tan^2 \left(\dfrac{\pi}{4} + \theta \right)$.

16. Show that $\cos^8 \theta - \sin^8 \theta = \cos 2\theta - \frac{1}{4} \sin 2\theta \sin 4\theta$.

17. $\cos \theta = \dfrac{\cos u - e}{1 - e \cos u}$, prove that

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}.$$

18. Given that $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$, find $\cos 18^\circ$.

19. Give that $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$, find $\cos 9^\circ$.

20. Show that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ and determine $\cos 36^\circ$.

Prove that $\sin 36^\circ \sin 72^\circ \sin 144^\circ = \frac{5}{16}$.

21. Prove that

$$(\sin A - \sin B)^2 + (\cos A - \cos B)^2 = 4 \sin^2 \frac{A-B}{2}.$$

22. Show that $\operatorname{cosec} A + 2 \operatorname{cosec} 2A = \sec A \cot \frac{A}{2}$.

23. Given $\tan \alpha = 1 + \sqrt{2}$, find $\cos 2\alpha$.

24. Find the circular functions of 30° and 15°. Show that $\tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ = 1$.

25. Express $\cos 5A$ in terms of $\cos A$. Hence or otherwise find the value of $\cos 18^\circ$.

26. Show that $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4 \cos 2\theta \cos 4\theta$.

27. Show that $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$.

28. Show that $\cos 5A = 5 \cos A - 20 \cos^3 A + 16 \cos^5 A$.

29. If θ_1 and θ_2 be two distinct angles satisfying the equation $a \cos 2\theta + b \sin 2\theta = c$, show that

$$\cos^2\theta_1 + \cos^2\theta_2 = \frac{a^2 + ac + b^2}{a^2 + b^2}.$$

30. If $x^2 + y^2 = 1$, show that $(3x - 4x^3)^2 + (3y - 4y^3)^2 = 1$.

31. If θ_1, θ_2 be two different angles satisfying the equation $a \cos \theta + b \sin \theta = c$, show that

$$\tan \frac{\theta_1}{2} + \tan \frac{\theta_2}{2} = \frac{2b}{c+a} \text{ and } \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{c-a}{c+a}.$$

32. If $\cos 4\theta = n$, find from this an eqaution for $\sin \theta$ and apply it to find $\sin \frac{\pi}{8}$.

33. Eliminate θ from the equation $a \cos 2\theta = b\ b \sin \theta$ and $c \sin 2\theta = d \cos \theta$.

35. Prove that $16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 1$.

(C. U.)

35. Show that $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$. (B. U.)

36. If $\cos A = \frac{40}{41}$ and $\cos B = \frac{60}{61}$ and A and B are positive and acute, show that $\sin^2\left(\frac{A-B}{2}\right) = \frac{1}{41 \times 61}$. (B. U.)

37. Show that (*i*) $\sin A = \sin(36^\circ + A) - \sin(36^\circ - A) + \sin(72^\circ - A) - \sin(72^\circ + A)$.

(*ii*) $\cos A = \sin(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A)$. (B. U.)

38. If $\cos A = \frac{a^2-1}{a^2+1}$ and $\cos B = \frac{b^2-1}{b^2+1}$ where A and B are positive acute angles, show that

$$\sin^2\left(\frac{A-B}{2}\right) = \frac{(a-b)^2}{(a^2+1)(b^2+1)}. \quad \text{(B. U.)}$$

39. If $\cos\theta = \frac{\cos\phi - C}{1 - C\cos\phi}$, then $C = \dfrac{\tan^2\frac{\theta}{2} - \tan^2\frac{\phi}{2}}{\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2}}$

(B. U.)

40. Show that $\tan 15^\circ + \cot 15^\circ = 4$.

Show that $\sin\frac{\pi}{24} = \frac{1}{4}(1+\sqrt{2}-\sqrt{3})\sqrt{2-\sqrt{2}}$. (B. U.)

41. If the equation $a\cos\theta + b\sin\theta = c$ has two distinct roots of θ to be α, β each less than 2π, then

$$\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}. \quad \text{(B. U.)}$$

42. If α, β be two values of θ in $a\tan\theta + b\sec\theta = c$,

then $\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$ and $\tan(\alpha-\beta) = \frac{\pm 2b(a^2+c^2-b^2)^{\frac{1}{2}}}{2b^2-c^2-a^2}$.

43. Prove that $4(\sin 24^\circ + \cos 6^\circ) = \sqrt{3} + \sqrt{15}$

$\sin 27^\circ = \frac{1}{8}(2\sqrt{5+\sqrt{5}} - \sqrt{10} + \sqrt{2})$.

44. Show that $\tan\frac{\pi}{8}$ is a root of $t^2 + 2t - 1 = 0$.

[Hint. Let $\theta = \frac{\pi}{8}$ so that $2\theta = \frac{\pi}{4}$ and $\therefore \tan 2\theta = 1$

or $\frac{2\tan\theta}{1-\tan^2\theta} = 1$ or $t^2 + 2t - 1 = 0$ where $t = \tan\theta$].

45. Show that $\sin\frac{\pi}{14}$ is a root of the equation

$$8x^3-4x^2-4x+1=0.$$

[Hint. Let $\theta=\frac{\pi}{14}$ so that $7\theta=\frac{\pi}{2}$ or $4\theta=\frac{\pi}{2}-3\theta$

therefore $\sin 4\theta=\sin\left(\frac{\pi}{2}-3\theta\right)=\cos 3\theta$

or $4\sin\theta\cos^3\theta-4\cos\theta\sin^3\theta=4\cos^3\theta-3\cos\theta$].

46. If $\tan\theta+\tan 2\theta=\tan 3\theta$, show that θ must be a multiple of $60°$ or $90°$.

47. In any triangle ABC, prove that

$$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)-\sin(A+B+C)=4\sin A\sin B\sin C.$$

48. Prove that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1,$$

where $A+B+C=\pi$.

If $A+B+C=180°$, show that

49. $$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}.$$

50. $$\sin A\sin B\sin C=\sin A\cos B\cos C+\sin B\cos A\cos C+\sin C\cos A\cos B.$$

51. $$\sin 4nA+\sin 4nB+\sin 4nC=-4\sin 2nA\sin 2nB\sin 2nC.$$

52. $$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}.$$

53. $$\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C.$$

54 If $A+B+C+D=2\pi$, show that

$$\sin A+\sin B+\sin C+\sin D=4\sin\frac{A+B}{2}\sin\frac{A+C}{2}\sin\frac{A+D}{2}.$$

55. Prove that if $c^2<a^2+b^2$ the equation

$$a\cos\theta+b\sin\theta=c$$

is satisfied by two and only two values of θ between 0 and 2π. If these values are θ_1, and θ_2 prove that

$$\tan\theta_1+\tan\theta_2=\frac{2ab}{c^2-b^2}.$$

56. **Show that**

$$\cot A \cot B \cot C = \cot A + \cot B + \cot C - \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C,$$ when A, B, C, are angles of a triangle.

57. If $x+y+z=xyz$, show that

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz.$$

58. $$\sin^2\frac{A+B}{2}\cos^2\frac{A-B}{2}+\cos^2\frac{A+B}{2}\sin^2\frac{A-B}{2}$$
$$=1-\tfrac{1}{2}\cos^2 A-\tfrac{1}{2}\cos^2 B.$$

59. If $\tan^2\frac{\theta}{2}=\tan^2\frac{A}{2}\tan^2\frac{B}{2}$, then show that

$$\cos\theta=\frac{\cos A+\cos B}{1+\cos A\cos B}.$$

60. Express $\cos\theta$ in terms of $\tan\frac{\theta}{2}$.

If $\tan^2\frac{\theta}{2}=\dfrac{1-e+(1+e)\tan^2\frac{\phi}{2}}{1+e+(1-e)\tan^2\frac{\phi}{2}}$, find $\cos\theta$ in terms of e and $\cos\phi$.

61. If θ_1 and θ_2 are the roots of the equation $a\cos\theta+b\sin\theta=c$, prove that

$$\frac{a}{\cos\frac{\theta_1+\theta_2}{2}}=\frac{b}{\sin\frac{\theta_1+\theta_2}{2}}=\frac{c}{\cos\frac{\theta_1-\theta_2}{2}}.$$

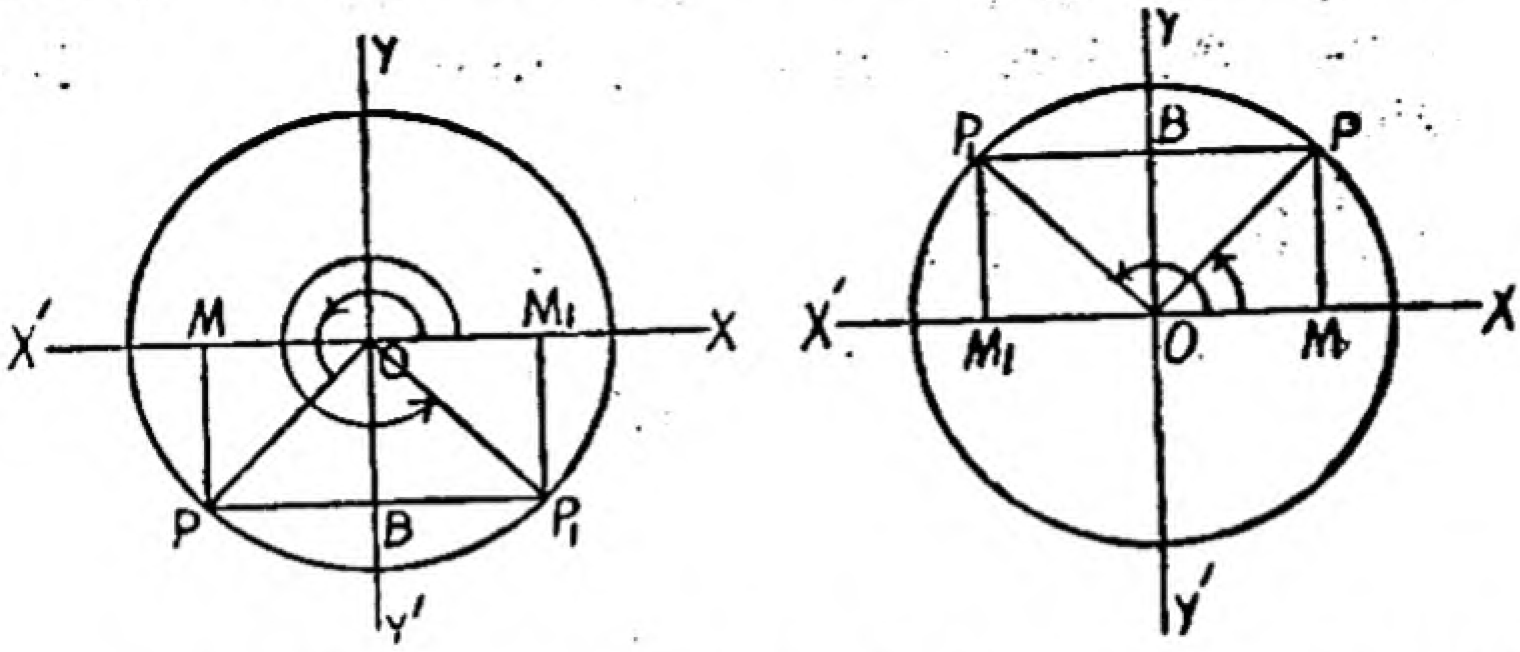
CHAPTER IX

INVERSE CIRCULAR FUNCTIONS

55. So far we have discussed the values of Circular Functions for a given value of the angle. In the present chapter the inverse process will be discussed, *viz.*, that given the value of a circular function, to find the value of its argument. Thus, for example, if $\sin\theta=\frac{1}{2}$, we can conclude that $\theta=30°$. It will be seen that $\theta=150°$ also satifies $\sin\theta=\frac{1}{2}$. We shall however see that for a given value of the trigonometrical ratio its argument is many valued.

56. *To construct an angle lying between 0 and 360° whose sine or cosecant is given.*

Let the given sine be k, k being positive or negative.



Take two lines X'OX and Y'OY cutting at right angles. With O as centre and unity as radius, describe a circle. Cut off a length OB, equal to k in magnitude, along OY if k is positive, or along OY' if k is negative. Draw $PBP_1 \parallel X'OX$ cutting the circle in P and P_1. Then $\angle XOP$ and XOP_1 are the angles having the given sine.

$$\text{For } \sin XOP=\frac{MP}{OP}=\frac{OB}{OP}=k\text{; and}$$

$$\sin XOP_1=\frac{M_1P_1}{OP_1}=\frac{OB}{OP_1}=k.$$

Cor. It follows that there are two angles lying between 0° and 360° which have a given sine.

Note 1.—Observe that the construction fails if k is numerically greater than unity, which is otherwise obvious.

Note 2.—If the cosecant of an angle be given, then its sine is known and a similar construction holds.

Notation. An angle whose sine is k is denoted by $\sin^{-1}k$, which is read as 'inverse sine k.' Similarly $\operatorname{cosec}^{-1}k$ denotes an angle whose cosecant is k. Therefore if $y=\sin^{-1}x$, then $x=\sin y$; and if $y=\operatorname{cosec}^{-1}x$, then $x=\operatorname{cosec} y$.

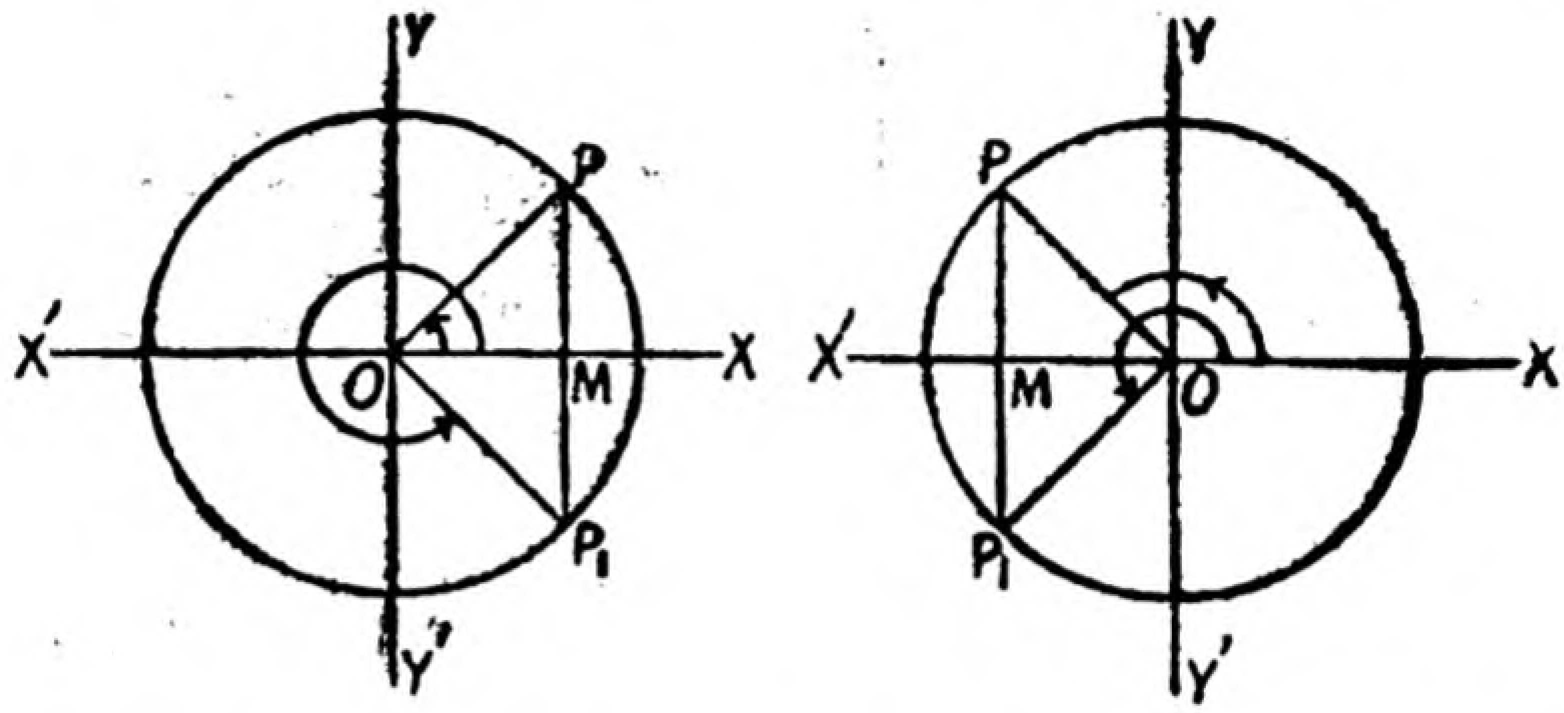
Notice that while $\sin x=\dfrac{1}{\operatorname{cosec} x}$, $\sin^{-1}x$ is not equal to $\dfrac{1}{\operatorname{cosec}^{-1}x}$.

Ex. Construct an angle whose sine is
(i) $\frac{1}{2}$. (ii) $-\frac{1}{3}$. (iii) $\frac{1}{5}$. (iv) $-\frac{2}{3}$.

57. ***To construct an angle lying between 0° and 360° whose cosine or secant is given.***

Let the given cosine be k, k being positive or negative.

Take two lines X'OX and Y'OY intersecting at right angles. With O as centre and unity as radius describe a



circle. Cut off a length OM equal to k, in magnitude, along OX if k is positixe and along OX' if k is negative. Draw $PMP_1 = Y'OY$ cutting the circle in P and P_1. Then $\angle XOP$ and $\angle XOP_1$ are the angles having the given cosine. For,

$$\cos XOP = \frac{OM}{OP} = k\ ; \text{ and } \cos XOP_1 = \frac{OM}{OP_1} = k.$$

Cor. It follows that there two angles lying between 0° and 360° which have a given cosine.

Note 1.—Observe that the construction fails if k is numerically greater than unity, which is otherwise obvious.

Note 2.—If the secant of an angle be given, then its cosine is known and a similar construction holds.

Notation. $\cos^{-1}k$ denotes an angle whose cosine is k and $\sec^{-1}k$ denotes an angle whose secant is k. Therefore if $y = \cos^{-1}x$, then $x = \cos y$, and if $y = \sec^{-1}x$ then $x = \sec y$.

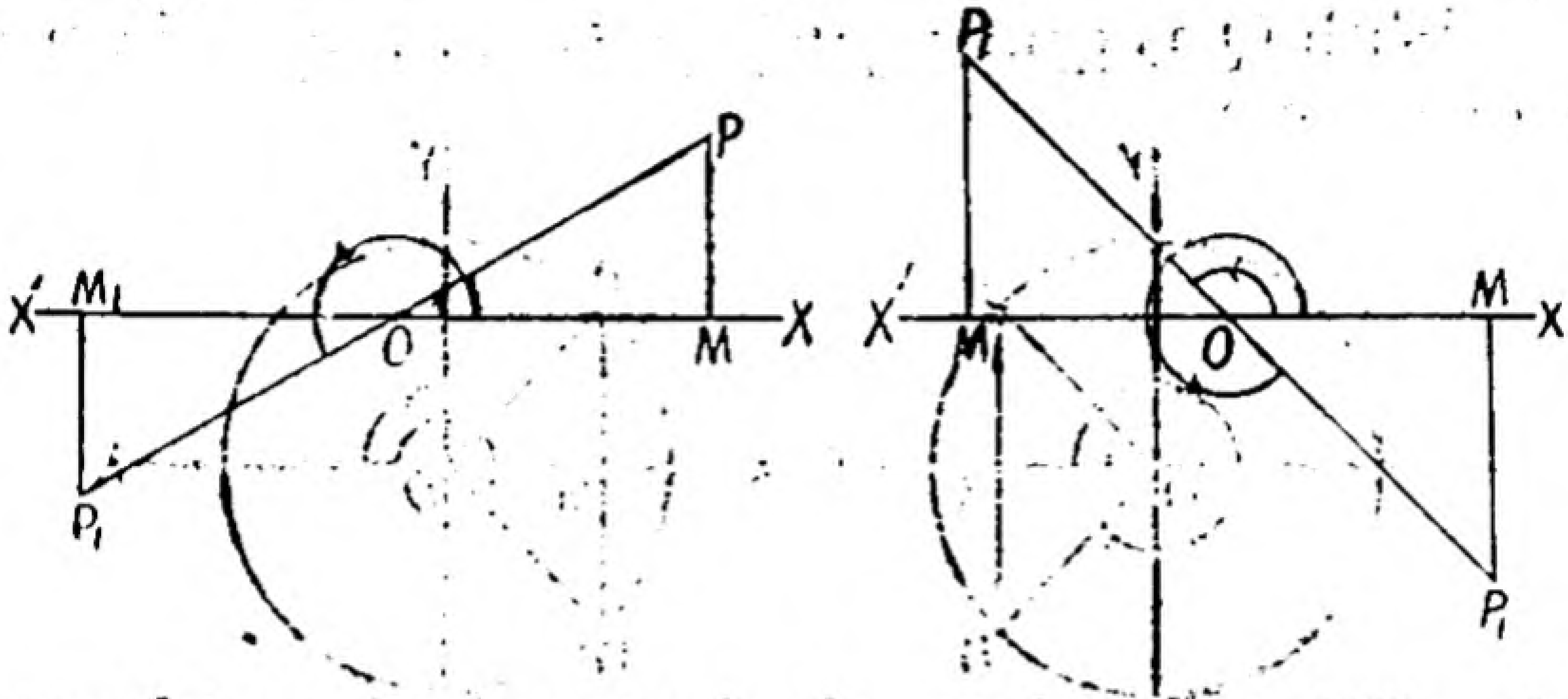
Notice that while $\cos x = \dfrac{1}{\sec x}$; $\cos^{-1}x$ is not equal to

$\dfrac{1}{\sec^{-1}x}$, but $\cos^{-1}x = \sec^{-1}\dfrac{1}{x}$, *e.g.*, $\cos^{-1}\frac{1}{2} = 60°$; $\sec^{-1} 2 = 60°$.

Ex. Construct an angle whose cosine is (*i*) $\frac{2}{3}$, (*ii*) $-\frac{3}{4}$.

58. ***To construct an angle whose tangent or cotangent is given.***

Let the given tangent be k, k being positive or negative. Take a straight line X′OX. Cut off lengths OM and



OM_1 equal to unity in magnitude, and along OX and OX′ respectively. At M and M_1 draw MP and M_1P_1 at right angles to X′OX, and equal to k in magnitude upwards and downwards respectively, if k is positive, and downwards and upwards if k is negative. Then ∠XOP and XOP_1 are the required angles.

$$\text{For } \tan XOP = \frac{MP}{OM} = k \text{ and } \tan XOP_1 = \frac{M_1P_1}{OM_1} = k.$$

Cor. It follows that there are two angles lying between 0° and 360° having a given tangent.

Note 1.—Observe that the construction never fails, which is otherwise obvious.

Note 2.—If the cotangent of an angle be given, then its tangent is known and a similar construction holds

Notation. $\tan^{-1}k$ denotes an angle whose tangent is k; and $\cot^{-1}k$ denotes an angle whose cotangent is k, so that if $y=\tan^{-1}x$, then $x=\tan y$, and if $y=\cot^{-1}x$, then $x=\cot y$; *e.g.*, if $\tan 45°=1$ then $\tan^{-1} 1=45°$.

Notice that while $\tan x=\dfrac{1}{\cot x}$, $\tan^{-1}x$ is not equal to $\dfrac{1}{\cot^{-1}x}$.

Ex. Construct an angle whose tangent is (i) $\frac{5}{4}$ (ii) $-\frac{8}{5}$.

Note 1.—The functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sec^{-1}x$ are called six inverse circular functions.

Note 2.—It may be clearly understood that $\sin^{-1}x$ means an angle whose sine is x and it does not mean $\frac{1}{\sin x}$.

Note 3.—It should be noted that $\sin^{-1}(\sin\theta)=\theta$.

EXERCISE XX

Find the values of the following angles which lie between 0° and 90° :—

(*i*) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. (*ii*) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$. (*iii*) $\tan^{-1}\frac{1}{\sqrt{3}}$.

(*iv*) $\cot^{-1}(1)$. (*v*) $\sin^{-1}(\cdot 8103)$. (*vi*) $\tan^{-1}(\cdot 2836)$.

2. Find the values of the following angles which lie between 0° and 360° :—

(*i*) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. (*ii*) $\tan^{-1}(\sqrt{3})$. (*iii*) $\operatorname{cosec}^{-1}(-2)$.

3. Solve the following equations, getting numerically the least values of θ :—

(*i*) $\sin^2\theta=1$. (*ii*) $2\cot^2\theta=\operatorname{cosec}^2\theta$.

(*iii*) $2\sin^2\theta+\sqrt{3}\cos\theta+1=0$. (*iv*) $\tan 5\theta=\cot 2\theta$.

4. If $\cos(A-B)=\frac{1}{2}$ and $\sin(A+B)=\frac{1}{2}$, ~~find~~ find the smallest positive values of A and B.

59. *Some Important Relations between Inverse Circular functions* :—

(*a*) (*i*) $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x}$. (*ii*) $\cos^{-1}x=\sec^{-1}\frac{1}{x}$,

(*iii*) $\tan^{-1}x=\cot^{-1}\frac{1}{x}$.

(*i*) Let $y=\sin^{-1}x$

$\therefore\quad x=\sin y$, *i e.*, $\frac{1}{x}=\operatorname{cosec} y$ or $\operatorname{cosec}^{-1}\frac{1}{x}=y$.

This proves the result. Now (*ii*) and (*iii*) also follow exactly the same way.

(*b*) (*i*) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

(*ii*) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}[xy-\sqrt{(1-x^2)(1-y^2)}]$

(*iii*) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\frac{x\pm y}{1\mp xy}$.

(*i*) Let $\sin^{-1}x=\theta$; $\sin^{-1}y=\phi$. $\therefore$ $x=\sin\theta$ and $y=\sin\phi$

$\therefore$ Left-hand side expression $=\theta+\phi$.

Right-hand side $=\sin^{-1}(\sin\theta\sqrt{1-\sin^2\phi}+\sin\phi\sqrt{1-\sin^2\theta})$

$=\sin^{-1}(\sin\theta\cos\phi+\sin\phi\cos^1\theta)$

$=\sin^{-1}\sin(\theta+\phi)$

$=\theta+\phi.$

Hence the result

(*ii*) This also follows as above.

(*iii*) Here let

$\tan^{-1}x=\theta,\ \tan^{-1}y=\phi$

$\therefore\ x=\tan\theta,\ y=\tan\phi.$

$\therefore\quad \tan^{-1}x\pm\tan^{-1}y=\theta\pm\phi.$

and $\tan^{-1}\left(\dfrac{x\pm y}{1\mp xy}\right)=\tan^{-1}\left(\dfrac{\tan\theta\pm\tan\phi}{1\pm\tan\theta\tan\phi}\right)$

$=\tan^{-1}\tan(\theta\pm\phi),$

$=\theta\pm\phi$

EXERCISE XXI

1. Show that :—

(*i*) $\tan^{-1}\dfrac{2x}{1-x^2}=2\tan^{-1}x.$ (*ii*) $\cos^{-1}(1-2x^2)=2\sin^{-1}x.$

(*iii*) $\sin^{-1}(3x-4x^3)=3\sin^{-1}x.$

(*iv*) $\tan^{-1}\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}=\dfrac{\theta}{2}.$

Show that :—

2. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}.$

3. $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}.$

4. $\cos^{-1}x=2\sin^{-1}\sqrt{\dfrac{1-x}{2}}=2\cos^{-1}\sqrt{\dfrac{1+x}{2}}.$

5. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\dfrac{\pi}{4}.$

6. $\tan^{-1}n+\cot^{-1}(n+1)=\tan^{-1}(n^2+n+1).$

7. Solve the equation $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}.$

[**Hint.** $\tan\dfrac{\pi}{4}=1=\tan(\tan^{-1}2x+\tan^{-1}3x)=\dfrac{2x+3x}{1-2x\times 3x}.$

$\therefore\quad 5x=1-6x^2$ or $6x^2+5x-1=0,$ or $x=-1$ or $\frac{1}{6}$.]

8. Draw the graph of $\cos^{-1}x.$

[Hint :—Let $y=\cos^{-1}x\ \therefore\ x=\cos y$ and thus the graph bears the same relation to OY that the curve in Art. 35 bears to OX.]

9. Show how to construct geometrically an angle whose cosine is a known negative quantity. Divide a rt. angle into two parts so that the cosine of one part may be double that of the other part. Give geometrical construction,

(B. U.)

Trigonometrical Equations

60. To find the general expression for all angles whose sine is zero.

We have to solve the equation $\sin\theta=0$.

The sin of an angle is zero only when the revolving line coincides with OX or OX′. Hence when $\sin\theta=0$, θ must be 0, or $\pm\pi$, or $\pm 2\pi$ or $\pm 3\pi$ and so on.

All these values are included in $\theta=n\pi$ where n is a positive or negative integer or zero,

Hence **if $\sin\theta=0$, then $\theta=n\pi$,**

where n is a positive or negative integer or zero.

61. To find the general expression for all angles whose cosine is zero.

We have to solve the equation $\cos\theta=0$.

The cosine of an angle is zero when the revolving line coincides with OY or OY′. Hence when $\cos\theta=0$, θ must be equal to

$$\pm\frac{\pi}{2}, \text{ or } \pm\frac{3\pi}{2} \text{ or } \pm\frac{5\pi}{2} \text{ and so on.}$$

All these values are included in the expression,

$\theta=(2n+1)\dfrac{\pi}{2}$, where n is a positive or negative integer or zero

Hence **if $\cos\theta=0$, $\theta=(2n+1)\dfrac{\pi}{2}$**

where n is a positive or negative integer or zero.

62. To find the general expression for all angles having a given sine.

Let α measured in radians be the smallest positive or negative angle having the given sine, and θ any angle having the same sine.

We have then to find the most general value of θ which satisfies the equation

$$\sin\theta=\sin\alpha$$

i.e., $\sin\theta-\sin\alpha=0$,

or $2\cos\dfrac{\theta+\alpha}{2}\sin\dfrac{\theta-\alpha}{2}=0$

$\therefore$ either $\cos\dfrac{\theta+\alpha}{2}=0$

which gives

$\dfrac{\theta+\alpha}{2}=(2p+1)\dfrac{\pi}{2}$

i.e., $\theta=(2p+1)\pi-\alpha$ (1)

or $\sin\dfrac{\theta-\alpha}{2}=0$

which gives

$\dfrac{\theta-\alpha}{2}=r\pi$

i.e., $\theta=2r\pi+\alpha$. (2)

The expressions (1) and (2) are both included in $\theta=n\pi+(-1)^n\alpha$, where n is zero or a positive or negative integer. For when n is odd this expression agrees with (1) and when n is even it agrees with (2).

Another Method. Suppose α is the smallest positive angle which has the given sine. Then we want to find θ from the equation

$$\sin\theta=\sin\alpha.$$

Evidently one value of θ is α. To this value we may add any number of complete revolutions without changing its sine. Thus a more general value of θ is given by

$\theta=2p\pi+\alpha$ (p being 0 or an integer). (1)

Again $\therefore \sin\alpha=\sin(\pi-\alpha)$

$\therefore \sin\theta=\sin(\pi-\alpha)$.

Hence another value of θ is $\pi-\alpha$. To this also we may add any number of complete revolutions without altering its sine. Hence a value of θ more general than $\pi-\alpha$ is given by $\theta=2r\pi+\pi-\alpha$

$=(2r+1)\pi-\alpha$ (r being 0 or an integer) (2)

As before, the results (1) and (2) are both included in one single formula $\theta=n\pi+(-1)^n\alpha$,

n being 0 or an integer, positive or negative, even or odd.

Cor. *Since all angles which have the same sine have also the same cosecant, this last expression includes all angles which have the same cosecant as* α.

Ex. 1. Solve the equation $\sin\theta=\dfrac{1}{2}$.

$\sin\theta=\dfrac{1}{2}=\sin\dfrac{\pi}{6}$, $\therefore \theta=n\pi+(-1)^n\dfrac{\pi}{6}$.

Ex. 2. Solve the equation $\sin\theta = -\dfrac{\sqrt{3}}{2}$.

$$\sin\theta = -\sin\frac{\pi}{3} = \sin\left(-\frac{\pi}{3}\right)$$

$$\therefore\quad \theta = n\pi + (-1)^n\left(-\frac{\pi}{3}\right) \quad\text{or}\quad \theta = n\pi - (-1)^n\frac{\pi}{3}.$$

63. To find the general expression for all angles which have a given cosine.

Let α measured in radians be the smallest positive or negative angle having the given cosine and θ any other angle having the given cosine.

Then we have to solve the equation $\cos\theta = \cos\alpha$:

i.e., $$\cos\theta - \cos\alpha = 0$$

or $2\sin\dfrac{\theta+\alpha}{2}\sin\dfrac{\alpha-\theta}{2} = 0$, or $-2\sin\dfrac{\theta+\alpha}{2}\sin\dfrac{\theta-\alpha}{2} = 0$:

$\therefore$ either $\sin\dfrac{\theta+\alpha}{2} = 0$,

which gives

$$\frac{\theta+\alpha}{2} = p\pi$$

i.e., $\theta = 2p\pi - \alpha$. (1)

or $\sin\dfrac{\theta-\alpha}{2} = 0$,

which gives

$$\frac{\theta-\alpha}{2} = r\pi$$

i.e., $\theta = 2r\pi + \alpha$. (2)

The expressions **(1)** and (2) are both included in $=2n\pi \pm \alpha$, when n is zero or a positive or negative integer.

Another Method. Suppose α is the smallest positive angle which has the given cosine. Then we want to find θ from the equation $\cos\theta = \cos\alpha$.

Evidently one value of θ is α. To this value we may add any number of complete revolutions without changing its cosine. Thus a more general value of θ is given by

$\theta = 2p\pi + \alpha$ (p being 0 or an integer). (2)

Again, $\because \cos\alpha = \cos(-\alpha)$ $\therefore \cos\theta = \cos(-\alpha)$

Hence another value of θ is $-\alpha$. To this also we may add any number of complete revolutions without altering its cosine. Hence another more general value of θ is given by

$\theta = 2r\pi - \alpha$ (r being 0 or an integer)

As before, both these may be put in a brief form as

$$\theta = 2n\pi \pm \alpha.$$

n being 0 or an integer, positive or negative, even or odd.

Cor. *Since all angles which have the same cosine have also the same secant, this last expression includes all angles which have the same secant.*

Ex. 1. Solve the equation $\cos\theta=\frac{1}{\sqrt{2}}$.

$\cos\theta=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$ $\quad\therefore\ \theta=2n\pi\pm\frac{\pi}{4}$.

Ex. 2. Solve the equation $\cos\theta=-\frac{1}{2}$.

$\cos\theta=\cos\frac{2\pi}{3}$ $\quad\therefore\ \theta=2n\pi\pm\frac{2\pi}{3}$.

64. To find the general expression for all angles having a given tangent.

Let α be the smallest positive or negative angle having the given tangent and let θ be any angle having the given tangent.

Then we have to solve the equation $\tan\theta=\tan\alpha$.

We have $\frac{\sin\theta}{\cos\theta}-\frac{\sin\alpha}{\cos\alpha}=0$,

or $\quad\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha}=0$;

$\therefore\quad\sin\theta\cos\alpha-\cos\theta\sin\alpha=0$; or $\sin(\theta-\alpha)=0$.

Hence $\theta-\alpha=n\alpha$ i.e., (π) $\theta=n\alpha+\alpha$. π

where n is zero or a positive or a negative integer.

Another Method. Suppose α is the smallest positive angle which has the given tangent. Then we want to find θ from the equation $\tan\theta=\tan\alpha$.

Evidently one value of θ is α. To this value we may add any integral multiple of 2π without changing its tangent. Thus a more general value of θ is $\theta=2p\pi+\alpha$.

(p being 0 or an integer). (1)

Again $\because\ \tan\alpha=\tan(\pi+\alpha)$ $\quad\therefore\ \tan\theta=\tan(\pi+\alpha)$.

Hence another value of θ is $\pi+\alpha$. To this also we may add any integral multiple of 2π without altering its tangent.

Hence a value of θ more general than $\pi+\alpha$ is

$\theta=2r\pi+\pi+\alpha$

$\quad=(2r+1)\pi+\alpha$ $\qquad$ (r being 0 or integer). (2)

As before, both these are included in $\theta = n\pi + a$, n being 0 or an integer positive or negative, even or odd.

Cor. *Since all angles which have the same tangent have also the same cotangent, this last expression includes all angles which have the same cotangent as* a.

Ex. 1. Solve the equation $\tan\theta = \sqrt{3}$.

Here $\sqrt{3} = \tan\dfrac{\pi}{3}$

$\therefore \quad \tan\theta = \tan\dfrac{\pi}{3}; \quad \theta = n\pi + \dfrac{\pi}{3}$

Ex. 2. Solve the equation $\tan\theta = -1$.

$\tan\theta = -1 = \tan\left(-\dfrac{\pi}{4}\right) \quad \therefore \theta = n\pi - \dfrac{\pi}{4}$.

Ex. 3. Find the most general value of θ satisfying the following equations simultaneously :

$$(i)\ \cos\theta = \frac{1}{\sqrt{2}} \text{ and } (ii)\ \tan\theta = 1.$$

$$\cos\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \quad \therefore \quad \theta = 2n\pi \pm \frac{\pi}{4}, \qquad (1)$$

when n is an integer, positive or negative.

Again $\tan\theta = 1 = \tan\dfrac{\pi}{4} \qquad \therefore \theta = m\pi + \dfrac{\pi}{4}, \qquad (2)$

when m is an integer, positive or negative, even or odd.

Now (i) and (ii) are to be satisfied simultaneously therefore we have to select an answer for θ which is common to (1) and (2). Such an answer is $2k\pi + \dfrac{\pi}{4}$ where k is an integer.

Note.—The ordinary methods for solving algebraic equations are often used in solving trigonometrical equations.

Ex. 4. Find the general value of θ for which the following equations are simultaneously satisfied :

$$\cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \tan\theta = 1.$$

$$\cos\theta = -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4} \qquad \therefore \theta = 2n\pi \pm \frac{3\pi}{4}$$

which is of the form $(2p+1)\pi \pm \frac{\pi}{4}$.

Also $\tan\theta = 1 = \tan\frac{\pi}{4}$ $\therefore\ \theta = m\pi + \frac{\pi}{4}$.

The form of θ found in both these expressions is

$$\theta = (2k+1)\pi + \frac{\pi}{4}.$$

Otherwise thus: Let us consider angles lying between 0° and 360°. The equation $\cos\theta = -\frac{1}{\sqrt{2}}$ is satisfied for $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{4}$ and the equation $\tan\theta = 1$ is satisfied for $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

Therefore both the equations are satisfied for $\theta = \frac{5\pi}{4}$. Hence the most general solution is

$$\theta = 2k\pi + \frac{5\pi}{4} \qquad \text{or } \theta = (2k+1)\pi + \frac{\pi}{4}.$$

Ex. 5. Solve the equation $4\cos^2\theta - 4\sin\theta = 1$.

The equation can be written as

$$4(1-\sin^2\theta) - 4\sin\theta - 1 = 0$$

or $\quad 4\sin^2\theta + 4\sin\theta - 3 = 0.$

or $\quad (2\sin\theta + 3)(2\sin\theta - 1) = 0,$

$\therefore \quad \sin\theta = \frac{1}{2} \quad$ or $\quad \sin\theta = -\frac{3}{2}$.

But $\sin\theta = -\frac{3}{2}$ must be rejected, because $\sin\theta$ is never greater than unity numerically.

Hence $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$ $\quad\therefore\ \theta = n\pi + (-1)^n\frac{\pi}{6}$.

Ex. 6. Solve the equation $\sin\theta + \cos\theta = \sqrt{2}$.

Transposing, $\cos\theta = \sqrt{2} - \sin\theta$.

Squaring, we get

$$\cos^2\theta = 2 - 2\sqrt{2}\sin\theta + \sin^2\theta$$

or $\quad 1 - \sin^2\theta = 2 - 2\sqrt{2}\sin\theta + \sin^2\theta;$

$\therefore \quad 2\sin^2\theta - 2\sqrt{2}\sin\theta + 1 = 0,$

whence $\sin\theta = \frac{1}{\sqrt{2}}$.

Substituting in the original equation, we get $\cos\theta=\frac{1}{\sqrt{2}}$.

The equations $\sin\theta=\frac{1}{\sqrt{2}}$ and $\cos\theta=\frac{1}{\sqrt{2}}$ are to be satisfied simultaneously.

The first is satisfied for $\theta=n\pi+(-1)^n\frac{\pi}{4}$ and the second for $\theta=2m\pi\pm\frac{\pi}{4}$. The form common to the two is given by $\theta=2k\pi+\frac{\pi}{4}$.

EXERCISE XXII

Find the most general values of θ satisfying the equations :

1. $\sin\theta=\frac{1}{2}$. 2. $\sin\theta=-\frac{\sqrt{3}}{2}$. 3. $\sec\theta=\sqrt{2}$.

4. $\cos\theta=-\frac{1}{2}$. 5. $\tan\theta=1$. 6. $\tan\theta=-\sqrt{3}$.

7. $\cot\theta=-1$. 8. $\sin 2\theta=1$. 9. $\cos 3\theta=\frac{1}{2}$.

10. $\tan 5\theta=-\frac{1}{\sqrt{3}}$. 11. $\sin^2\theta=\frac{3}{4}$.

12. $\tan^2\theta=\frac{1}{3}$. 13. $\sec^2\theta=4$.

14. $\sin\theta=\cos\alpha$.

[*Hint.* $\sin\theta=\sin\left(\frac{\pi}{2}-\alpha\right)\ \therefore\ \theta=n\pi+(-1)^n\left(\frac{\pi}{2}-\alpha\right)$].

Find the most general value of θ satisfying the following equations simultaneously :

15. $\sin\theta=-\frac{1}{2}$ and $\tan\theta=\frac{1}{\sqrt{3}}$.

16. $\cot\theta=-\sqrt{3}$ and $\sin\theta=\frac{1}{2}$.

Solve the equations :

17. $\cos(A-B)=\frac{1}{2}$ and $\sin(A+B)=\frac{1}{2}$.

18. $\tan(A-B)=1$ and $\cos(A+B)=\frac{\sqrt{3}}{2}$.

19. $\tan(A+B+C)=\sqrt{3}$
$\tan(A-B+C)=1$
$\tan(A+B-C)=\frac{1}{\sqrt{3}}$.

Solve the equations :

20. $2\cos^2\theta-7\cos\theta+5=0$. 21. $\sec^4\theta-6\sec^2\theta+8=0$.

22. $\tan^2\theta-\sec\theta-1=0$. 23. $4\cos^2\theta-4\sin\theta-1=0$.

24. $3\tan^2\theta+2\sqrt{3}\tan\theta-3=0$.

24. $\cos^2 x+\sin x=1$. (P. U. 1945).

65. When different circular functions of the same angle θ or the multiples of θ are involved in the equation, we have sometimes to transform the equation.

Ex. Solve the equation

$$\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}.$$

Let $\sqrt{3}=r\sin\phi$ (*i*).
and $1=r\cos\phi$ (*ii*), where r is positive.
Squaring and adding (*i*) and (*ii*), we get

$$r^2=4, \text{ so that } r=2.$$

Dividing (*i*) by (*ii*), we have $\tan\phi=\sqrt{3}$, $\therefore\ \phi=\frac{\pi}{3}$.

The equation now becomes

$$r\sin\phi\cos\theta+r\cos\phi\sin\theta=\sqrt{2},$$

i.e., $r\sin(\phi+\theta)=\sqrt{2}$.

or $$2\sin\left(\frac{\pi}{3}+\theta\right)=\sqrt{2}$$

or $$\sin\left(\theta+\frac{\pi}{3}\right)=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}.$$

$$\therefore\quad \theta+\frac{\pi}{3}=n\pi+(-1)^n\frac{\pi}{4}.$$

Hence $\theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$.

Note.—**Notice that r is taken as positive and ϕ is taken so that it satisfies both (*i*) and (*ii*).**

Now we shall take the general case of which the above is only a particular one.

66. To solve the equation

$$a \cos \theta + b \sin \theta = c.$$

Put $\quad a = r \sin \phi,\ b = r \cos \phi,$

where r is positive.

Squaring and adding we get

$$r^2 = a^2 + b^2 \qquad \therefore \quad r = \sqrt{a^2+b^2}.$$

The angle ϕ is known from the equations

$$\sin \phi = \frac{a}{r}, \quad \cos \phi = \frac{b}{r} \qquad \text{and} \quad \therefore \ \tan \phi = \frac{a}{b}.$$

Observe that as r is positive, ϕ must be so taken that its sine has the same sign as a and its cosine the same sign as b.

The given equation therefore becomes

$$r \sin \phi \cos \theta + r \cos \phi \sin \theta = c$$

$$\sin (\theta + \phi) = \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}}.$$

Now let an angle α be found such that $\sin \alpha = \dfrac{c}{\sqrt{a^2+b^2}}$ which is possible only when c is not greater than $\sqrt{a^2+b^2}$.

$$\therefore \quad \sin (\theta + \phi) = \frac{c}{\sqrt{a^2+b^2}} = \sin \alpha.$$

Hence $\theta + \phi = n\pi + (-1)^n \alpha$,

i.e., $\quad \theta = n\pi + (-1)^n \alpha - \phi.$

Note.—Notice that the equation $a \cos \theta + b \sin \theta = c$ can also be solved by the substitution $a = r \cos \phi,\ b = r \sin \phi$.

Ex. Solve the equation $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$.

Let $\quad \sqrt{3} = r \cos \phi.$

$\quad 1 = r \sin \phi. \quad \therefore \quad r^2 = 4$ or $r = 2$.

Also $\tan \phi = \dfrac{1}{\sqrt{3}} = \tan \dfrac{\pi}{6} \quad \therefore \quad \phi = \dfrac{\pi}{6}.$

The equation becomes $r \cos \phi \cos \theta + r \sin \phi \sin \theta = \sqrt{2}$

or $\cos \left(\theta - \dfrac{\pi}{6} \right) = \dfrac{\sqrt{2}}{r} = \dfrac{1}{\sqrt{2}} = \cos \dfrac{\pi}{4}.$

Hence $\theta-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{4}$, or $\theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}$.

Note. Observe that this answer to the equation appears to be different from the answer which we got in the previous article. But it is easy to identify the two answers. In fact whenever we get two apparently different answers to the same equation by different methods, the two answers can always be identified.

By solving the above equations in two ways we got

$(i)\ \theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$, and $(ii)\ \theta=n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}$.

We shall show that (i) is the same as (ii).

When n is odd the answer (i) takes the form

$2k\pi+\frac{\pi}{4}-\frac{\pi}{3}$, *i.e.*, $2k\pi-\frac{\pi}{12}$.

When n is even, answer (i) takes the form

$(2m+1)\pi-\frac{\pi}{4}-\frac{\pi}{3}$, *i.e.*, $2m\pi+\pi-\frac{\pi}{4}-\frac{\pi}{3}$ or $2m\pi+\frac{5\pi}{12}$.

So that the first answer is of the form

$$2m\pi-\frac{\pi}{12} \quad\text{or}\quad 2m\pi+\frac{5\pi}{12}.$$

But $\frac{5\pi}{12}=\frac{\pi}{4}+\frac{\pi}{6}$ and $-\frac{\pi}{12}=-\frac{\pi}{4}+\frac{\pi}{6}$.

Hence the first answer can be put in the form

$$2n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}, \text{ which is the answer } (ii).$$

Solved Examples

Ex. 1. Solve the equation $\sin 4\theta-\sin 2\theta=\cos 3\theta$.

Sol. $\sin 4\theta-\sin 2\theta=2\cos 3\theta\sin\theta$;

$\therefore\ 2\cos 3\theta\sin\theta=\cos 3\theta$.

$\therefore\ \cos 3\theta\,(2\sin\theta-1)=0$.

$\therefore$ either $\cos 3\theta=0$ which gives $3\theta=(2n+1)\frac{\pi}{2}$

i.e., $\theta=\frac{(2n+1)\pi}{6}$ or $\sin\theta=\frac{1}{2}=\sin\frac{\pi}{6}$ which gives

$\theta=n\pi+(-1)^n\frac{\pi}{6}$.

Ex. 2. Solve the equation $\sin m\theta = \cos n\theta$.

First Method. $\sin m\theta = \sin\left(\frac{\pi}{2} - n\theta\right)$

$$\therefore \qquad m\theta = k\pi + (-1)^k\left(\frac{\pi}{2} - n\theta\right)$$

or $m\theta + (-1)^k n\theta = k\pi + (-1)^k\left(\frac{\pi}{2}\right)$ or $\theta = \dfrac{k\pi + (-1)^k \frac{\pi}{2}}{m + (-1)^k}$.

Second Method. $\cos\left(\frac{\pi}{2} - m\theta\right) = \cos n\theta$

$$\frac{\pi}{2} - m\theta = 2k\pi \pm n\theta, \quad \text{or} \quad \theta = \frac{\frac{\pi}{2} - 2k\pi}{m \pm n}.$$

Note.—It is easy to see that the two answers are of the same form.

Ex. 3. Show that the equation $a\cos\theta + b\sin\theta = c$ can be solved by the substitution $\tan\frac{\theta}{2} = t$.

$$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{1 - t^2}{1 + t^2} \text{ and } \sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{2t}{1 + t^2}.$$

$\therefore$ the equation becomes

$$a\left(\frac{1 - t^2}{1 + t^2}\right) + b\left(\frac{2t}{1 + t^2}\right) = c \quad \text{or} \quad t^2(a + c) - 2bt + c - a = 0.$$

This gives two values of t or $\tan\frac{\theta}{2}$ from which θ can be found.

The solution is possible only when

$$b^2 - (c - a)(c + a) > 0,$$

i.e., $$a^2 + b^2 > c^2.$$

Note.—This method is convenient for numerical cases.

Solving the quadratic for t we get

$$t = \tan\frac{\theta}{2} = \frac{2b \pm \sqrt{4b^2 - 4(c^2 - a^2)}}{2(a + c)}$$

$$= \frac{b \pm \sqrt{a^2 + b^2 - c^2}}{a + c}$$

$$=\tan\alpha \text{ or } \tan\beta \quad \therefore \tan\frac{\theta}{2}=\tan\alpha$$

$$\therefore \quad \frac{\theta}{2}=n\pi+\alpha \quad \text{or} \quad \theta=2n\pi+2\alpha \text{ and } \tan\frac{\theta}{2}=\tan\beta$$

$$\therefore \quad \frac{\theta}{2}=n\pi+\beta \quad \text{or} \quad \theta=2n\pi+2n\pi\beta.$$

Hence $\theta=2n\pi+2\alpha$ or $2n\pi+2\beta$, where α and β are the least angles whose tangents are

$$\frac{b+\sqrt{a^2+b^2-c^2}}{a+c} \text{ and } \frac{b-\sqrt{a^2+b^2-c^2}}{a+c}.$$

Ex. 4. Solve the equation

$$3\cos^2\theta-2\sqrt{3}\sin\theta\cos\theta-3\sin^2\theta=0.$$

Dividing throughout by $\cos^2\theta$, we have

$$3-2\sqrt{3}\tan\theta-3\tan^2\theta=0,$$

$$\tan\theta=-\sqrt{3} \quad \text{or} \quad \frac{1}{\sqrt{3}}$$

$$\therefore \quad \theta=n\pi-\frac{\pi}{3} \quad \text{or} \quad \theta=n\pi+\frac{\pi}{6}.$$

EXERCISE XXIII

Solve the equations :

1. $\sin\theta+\cos\theta=\sqrt{2}.$
2. $\sin\theta-\cos\theta=\sqrt{2}.$
3. $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}.$
4. $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}.$
5. $\cos x+\sqrt{3}\sin x=2.$
6. $\cos 3x+\sin 3x=\frac{1}{\sqrt{2}}.$
7. $2\sqrt{2}\sin\theta\cos\theta=1.$
8. $\sin 2\theta=\sin 3\theta.$
9. $\sin m\theta=\sin n\theta.$
10. $\cos m\theta+\cos n\theta=0.$
11. $\tan m\theta=\tan n\theta.$
12. $\tan m\theta=\cot n\theta.$
13. $\sin\theta+\sin 3\theta+\sin 5\theta=0.$
14. $\sin 5\theta\cos 3\theta=\sin 9\theta\cos 7\theta.$
15. $\sin 3\theta=8\sin^3\theta.$
16. $\cos^2\theta-\cos\theta\sin\theta-\sin^2\theta=1.$
17. $2\sin^4\theta+\cos^4\theta=1.$
18. $\tan(\pi\cot\theta)=\cot(\pi\tan\theta).$
19. $\sin^2\theta=\sin^2\alpha.$ (B.U.)
20. $\sin 3x+\sin 2x+\sin x=0.$ (P. U. 1945)

Formulae on Chapter IX

1. $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x}.$ 2. $\cos^{-1}x=\sec^{-1}\frac{1}{x}.$

3. $\tan^{-1}x = \cot^{-1}\dfrac{1}{x}$.

4. $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$.

5. $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{(1-x^2)(1-y^2)}]$

6. $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\dfrac{x \pm xy}{1 \mp xy}$.

7. If (*a*) $\sin\theta = 0$, then $\theta = n\pi$

(*b*) $\cos\theta = 0$, then $\theta = (2n+1)\dfrac{\pi}{2}$.

(*c*) $\sin\theta = \sin\alpha$ then $\theta = n\pi + (-1)^n a$.

(*d*) $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$,

(*e*) $\tan\theta = \tan\theta$, then $\theta = n\pi + \alpha$.

REVISION QUESTIONS VI

1. If θ is an acute angle, find its value from the equation $3\tan\theta + \cot\theta = 5\operatorname{cosec}\theta$.

2. If $\tan x = 2 - \sqrt{3}$, find the value of x in radians.

3 If $\sin(x+y)\cos z = \sin(x+z)\cos y$, show that $y-z$ is a multiple of π or x an odd multiple of $\dfrac{\pi}{2}$.

4. Find the general expression for all angles having a given sine.

Given $\sin A = \frac{1}{2}$ find the general value of A, and also find the four least positive values of A.

Solve the equations :

5. $\sin^3\theta + \cos^3\theta = 0$.

6. $\tan\left(\dfrac{\pi}{4} - \theta\right) + \cot\left(\dfrac{\pi}{4} - \theta\right) = 4$..

7. $\tan\left(\dfrac{\pi}{4} + \theta\right) = 3\tan\left(\dfrac{\pi}{4} - \theta\right)$.

8. $\cos 2\theta = 2\cos^2\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

9. $\sin A + \cos A = \sqrt{2}$. 10. Solve $\sin 4\theta = \dfrac{\sqrt{3}}{2}$.

Find all the values of $\theta < 180°$ which satisfy this equation.

11. In an examination it was required to solve the equation $\sin\theta = -\frac{1}{2}$. One candidate found the answer to be $n\pi - (-1)^n \frac{\pi}{6}$ and another $n\pi + (-1)^n \frac{7\pi}{6}$.

Explain why both the answers are correct.

12. Solve $\sin\theta = \cos 2\theta$ in two different ways and identify the two answers.

13. Solve $\sin 7\theta - \sin\theta = \sin 3\theta$.

14. Solve $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$.

15. Solve $\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta$.

16. Find the solution of $\tan^2 x + \cot^2 x = 2$, x lying between 0° and 180°.

17. Prove that

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}].$$

18. Find x if

$$\tan^{-1}\frac{x}{1+x} + \tan^{-1}\frac{x}{1-x} = \tan^{-1}2. \qquad \text{(B. U.)}$$

CHAPTER X

RELATIONS BETWEEN THE SIDES AND THE ANGLES OF A TRIANGLE

The angles of triangle A,B,C are usually denoted by the capital letters A, B, C and the sides opposite to these angles are respectively denoted by $a, b, c,$

67. The Sine Formula.

To prove that in any triangle ABC,

$$\frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}}$$

i. e., the sines of the angles are proportional to the opposite sides.

Let ABC be the △ and let one of the angles, say B be acute ; C may then be acute, obtuse or right.

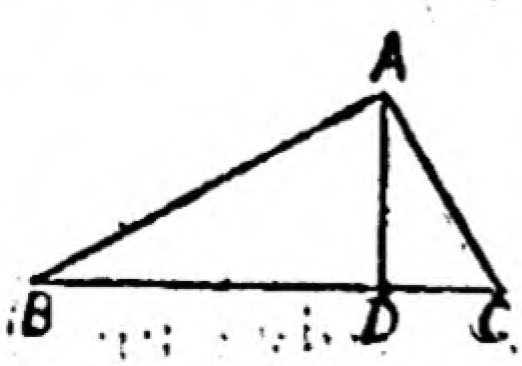


Fig. 1.

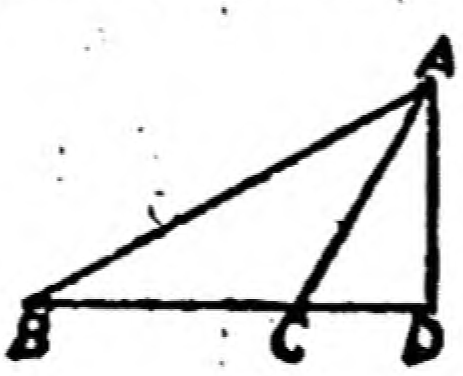


Fig. 2.

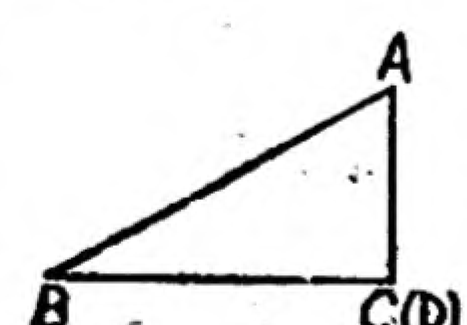


Fig. 3.

Draw $AD \perp BC$ or BC produced.

Then $\frac{DA}{BA}=\sin B$. $\therefore$ $DA=c \sin B$. ......(*i*)

If C is acute as in Fig. (1), $\frac{DA}{CA}=\sin C$

If C is obtuse as in (Fig. 2), $\frac{DA}{CA}=\sin \angle ACD$

$$=\sin(\pi-C)=\sin C.$$

If C is right as in Fig. (3), $\frac{DA}{CA}=1=\sin C$.

Hence in each case, $DA=b \sin C$. ......(*ii*)

From (*i*) and (*ii*), we have $b \sin C=c \sin B$.

$$\therefore \quad \frac{b}{\sin B}=\frac{c}{\sin C}.$$

Similarly $\frac{a}{\sin A}=\frac{c}{\sin C}$ $\quad\therefore\quad$ $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$

This is known as the **Sine Formula.**

Ex. 1. In any triangle ABC prove that
$a \cos A+b \cos B=c \cos(A-B)$.

Sol. In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \text{ (say)}$$

$\therefore$ $a=k \sin A,\ b=k \sin B,\ c=k \sin C$.

$\therefore$ $a \cos A+b \cos B=k \sin A \cos A+k \sin B \cos B.$

$$=\frac{k}{2}(2 \sin A \cos A+2 \sin B \cos B)$$

$$=\frac{k}{2}(\sin 2A+\sin 2B)$$

$$=\frac{k}{2}\times 2 \sin(A+B)\cos(A-B)$$

$$=k \sin(A+B)\cos(A-B)$$

$$=k \sin C \cos(A-B); \quad \because A+B=\pi-C$$

$$=c \cos(A-B).$$

Ex. 2. In any triangle ABC prove that

$$\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}.$$

Sol. In any triangle ABC,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \text{ (say)}$$

Then $a=k\sin A$, $b=k\sin B$, $c=k\sin C$.

$$\therefore \quad \frac{b-c}{a}=\frac{k\sin B-k\sin C}{k\sin A}=\frac{\sin B-\sin C}{\sin A}$$

$$=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\cos\left(\frac{B+C+A}{2}-\frac{A}{2}\right)\sin\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}.$$

By cross multiplication, $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$.

Ex. 3. In a triangle ABC, if $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$, prove that the triangle is equilateral.

Sol. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$. ...... (*i*)

Also $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ (given) ......(*ii*)

From (*i*) and (*ii*) by multiplication,

$$\cot A=\cot B=\cot C.$$

$\therefore$ A=B=C. $\therefore$ Δ is equilateral.

Ex. 4. If a straight line be drawn bisecting the angle A of a triangle ABC to meet the opposite side in D, show that the segments of this side are

$$\frac{a\sin C}{\sin C+\sin B} \text{ and } \frac{a\sin B}{\sin C+\sin B}$$

$$\frac{BD}{DC}=\frac{BA}{AC}=\frac{c}{b}=\frac{\sin C}{\sin B}$$

$$\therefore \quad \frac{BD}{\sin C}=\frac{DC}{\sin B}=\frac{BD+DC}{\sin C+\sin B}=\frac{a}{\sin B+\sin C},$$

so that $BD=\frac{a\sin C}{\sin C+\sin B}$ and $DC=\frac{a\sin B}{\sin C+\sin B}$.

EXERCISE XXIV

In any triangle ABC show that

1. $\sin \dfrac{B-C}{2} = \dfrac{b-c}{a} \cos \dfrac{A}{2}$.

2. $c \sin \dfrac{A-B}{2} = (a-b) \cos \dfrac{C}{2}$.

3. $(c+a) \sin \dfrac{B}{2} = b \cos \dfrac{C-A}{2}$.

4. $a \sin A + b \sin B + c \sin C = \sqrt{a^2+b^2+c^2} \sqrt{\sin^2 A + \sin^2 B + \sin^2 C}$.

5. $\dfrac{a \sin (B-C)}{b^2-c^2} = \dfrac{b \sin (C-A)}{c^2-a^2} = \dfrac{c \sin (A-B)}{a^2-b^2}$.

68. Napier's Analogies.

To prove that

(*i*) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$.

(*ii*) $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$.

(*iii*) $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$.

As the proof is similar in all the three cases we here prove only (*i*).

From Sine Formula we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\therefore \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$=\cot\frac{B+C}{2}\tan\frac{B-C}{2}$$

$$=\cot\left(90^\circ-\frac{A}{2}\right)\tan\frac{B-C}{2}$$

$$=\tan\frac{A}{2}\tan\frac{B-C}{2}$$

$$\therefore \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

69. The Cosine Formula.

To prove that $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$.

Let ABC be the $\triangle$ and let one of the angles, say B, be acute; C may be then acute, obtuse, or a right angle.

[See figure, Art. 67.]

Draw AD $\perp$ BC or BC produced.

From Fig. (1), $AB^2=BC^2+AC^2-2BC.CD$.

But $\dfrac{CD}{b}=\cos C$, or $CD=b\cos C$.

$\therefore\quad c^2=a^2+b^2-2ab\cos C$.

From Fig. (2) $AB^2=BC^2+CA^2+2BC\,CD$.

But $\dfrac{CD}{b}=\cos ACD=\cos(\pi-C)=-\cos C$.

$\therefore\quad c^2=a^2+b^2-2ab\cos C$.

From Fig. (3) $AB^2=BC^2+CA^2=a^2+b^2-2ab\cos C$

($\because\ \cos C=\cos 90^\circ=0$).

Thus in all cases, $c^2=a^2+b^2-2ab\cos C$

$\therefore\qquad 2ab\cos C=a^2+b^2-c^2$

or $$\cos C=\frac{a^2+b^2-c^2}{2ab}.$$

Similarly $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$, and $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$.

These are known as **Cosine Formulae**.

Cor. *The square of any side of a triangle = the sum of the squares of the other two sides minus twice their product and cosine of included angle.*

Remember :—With usual notation, in a $\triangle$ ABC :—

$$a^2=b^2+c^2-2bc\cos A$$
$$b^2=c^2+a^2-2ac\cos B$$
$$c^2=a^2+b^2-2ab\cos C.$$

Ex. 1. In a triangle ABC, $A=60^\circ$; prove that
$$(a+b+c)(b+c-a)=3bc.$$

Sol. $\cos A=\dfrac{b^2+c^2-a^2}{2bc}=\cos 60^\circ=\dfrac{1}{2}.$

$\therefore\quad b^2+c^2-a^2=bc,$
$\therefore\quad b^2+c^2+2bc-a^2=3bc,$ or $(b+c)^2-a^2=3bc.$
$\therefore\quad (b+c+a)(b+c-a)=3bc.$

Ex. 2. If $2\cos B=\dfrac{\sin A}{\sin C}$, prove that the triangle is isosceles.

Since $2\cos B=\dfrac{\sin A}{\sin C}\quad\therefore\quad\dfrac{c^2+a^2-b^2}{ca}=\dfrac{a}{c},$

or $c^2+a^2-b^2=a^2$ or $c^2=b^2$
$\therefore\quad c=b.$

Hence the triangle is isosceles.

70. The Projection Formulae.

To prove that $a=b\cos C+c\cos B.$

See Figs. of Art. 67.

From Fig. (1), $BC=BD+DC,$

but $\dfrac{BD}{BA}=\cos B$ and $\dfrac{DC}{AC}=\cos C,$

so that $BD=c\cos B$ and $DC=b\cos C,$

$\therefore\quad a=c\cos B+b\cos C.$

From Fig. (2), $BC=BD-CD$

i.e., $a=c\cos B-b\cos\angle ACD$
$=c\cos B-b\cos(\pi-C)$
$=c\cos B+b\cos C.$

From Fig. (3), $BC=c\cos B$
$=c\cos B+b\cos C$
$(\because\ \cos C=\cos 90^\circ=0).$

Thus in all cases,

$$\mathbf{a=b\cos C+c\cos B}$$

Similarly $\mathbf{b=a\cos C+c\cos A},$

and $\mathbf{c=a\cos B+b\cos A}.$

Ex. 1. Deduce from the Sine Formulae (*a*) Cosine Formulae and (*b*) the Projection Formulae.

From the Sine Formulae

$$a=k \sin A,\ b=k \sin B,\ c=k \sin C.$$

(*a*) $\therefore \quad b^2+c^2-a^2=k^2\ (\sin^2 B+\sin^2 C-\sin^2 A)$

$$=k^2\ \{\sin^2 B+\sin (C+A) \sin (C-A)\}$$
$$=k^2\ [\sin^2 B+\sin B \sin (C-A)]$$
$$=k^2 \sin B\ [\sin (C+A)+\sin (C-A)]$$
$$=k^2 \sin B\ 2 \sin C \cos A$$
$$=k \sin B \cdot 2k \sin C \cos A$$
$$=2bc \cos A.$$

$$\therefore \quad \cos A=\frac{b^2+c^2-a^2}{2bc}.$$

(*b*) From $\sin A=\sin (B+C)=\sin B \cos C+\cos B \sin C$, we get by the sine formulae

$$\frac{a}{k}=\frac{b}{k} \cos C+\cos B \frac{c}{k},$$

whence $a=b \cos C+c \cos B$.

Ex. 2. Deduce from the projection Formulae (*a*) the Cosine Formulae (*b*) the Sine Formulae.

Given $a=b \cos C+c \cos B$ ...... (*i*)

$b=c \cos A+a \cos C$ ...... (*ii*)

$c=a \cos B+b \cos A.$ ......(*iii*)

(*a*) Multiplying (*i*) by $-a$, (*ii*) by b and (*iii*) by c and adding, we get

$$-a^2+b^2+c^2=(-ab \cos C-ac \cos B)+$$
$$(bc \cos A+ba \cos C)+(ac \cos B+bc \cos A)$$
$$=2bc \cos A$$

$\therefore \quad \cos A=\dfrac{b^2+c^2-a^2}{2bc}$; similarly the other two formulae follow.

(*b*) From $a=b \cos C+c \cos B$

and $b=c \cos A+a \cos C$, we get

$$a-b \cos C-c \cos B=0$$

and $-a \cos C+b-c \cos A=0.$

Solving for a, b and c, we have

$$\frac{a}{\cos C \cos A+\cos B}=\frac{b}{\cos B \cos C+\cos A}=\frac{c}{1-\cos^2 C}.$$

Now $\cos B = -\cos(A+C) = -\cos A \cos C + \sin A \sin C$
and $\cos A = -\cos(B+C) = -\cos B \cos C + \sin B \sin C$

$$\therefore \quad \frac{a}{\sin A \sin C} = \frac{b}{\sin B \sin C} = \frac{c}{\sin^2 C}.$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Ex. 3. To deduce from the Cosine Formulae (*a*) the Sine Formulæ and (*b*) the Projection Formulæ,

Cosine formulæ are $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$;

$\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

(*a*) $$\frac{a^2}{\sin^2 A} = \frac{a^2}{1-\left(\dfrac{b^2+c^2-a^2}{2bc}\right)^2}$$

$$= \frac{4a^2b^2c^2}{(2bc)^2-(b^2+c^2-a^2)^2}$$

$$= \frac{4a^2b^2c^2}{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}$$

$$= \frac{4a^2b^2c^2}{(a+b+a)(b+c-a)(c+a-b)(a+b-c)}$$

a symmetrical expression in a, b and c.

In the same way it follows that $\dfrac{b^2}{\sin^2 B}$ and $\dfrac{c^2}{\sin^2 C}$ also are equal to the same expression.

Hence $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, because $\sin A$, $\sin B$ and $\sin C$ are all positive.

(*b*) $$b \cos C + c \cos B = b\frac{a^2+b^2-c^2}{2ab} + c\frac{a^2+c^2-b^2}{2ac}$$

$$= \frac{2a^2}{2b} = a,$$

$\therefore\ a = b \cos C + c \cos B.$

Similarly it follows that

$$b=c\cos A+a\cos C \text{ and } c=a\cos B+b\cos A.$$

Ex. 4. In any triangle, prove that

$$\Sigma(b^2-c^2)\tan B\tan C=0.$$

Here dividing both the sides by tan A tan B tan C, we get

$$\Sigma\frac{(b^2-c^2)}{\tan A}=0, \qquad \text{or} \quad \Sigma\frac{(b^2-c^2)}{\sin A}\cos A=0 \qquad \ldots\ldots(i)$$

Now $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ etc., and $\sin A=ka$ etc.

$$\therefore \quad (i) \text{ is } \Sigma\frac{(b^2-c^2)(b^2+c^2-a^2)}{kabc}=0$$

i.e., $\Sigma(b^2-c^2)(b^2+c^2-a^2)=0$ which is [illegible]e.

Ex. 5. *Show that in a triangle A B C if D be the middle point of BC then $AB^2+AC^2=2(BD^2+AD^2)$*

(Median Theorem)

Here let angle $ADB=\theta$ $\therefore$ $\angle ADC=180°-\theta$.

Also $AB^2=AD^2+BD^2-2\,AD.BD\cos\angle ADC$

$=AD^2+BD^2-2AD.BD\cos\theta$ $\ldots\ldots\ldots(i)$

and $AC^2=AD^2+DC^2-2AD.DC\cos\angle ADC$

$=AD^2+DC^2+2AD.DC\cos\theta.$ $\ldots\ldots\ldots(ii)$

Adding (i) and (ii) and putting $DC=BD$ we get $AB^2+AC^2=2(AD^2+BD^2)$ which is the required result

EXERCISE XXV

In any triangle ABC show that :

1. $\cos\dfrac{B-C}{2}=2\sin\dfrac{A}{2}$, if $b+c=2a$.
2. $\dfrac{b^2-c^2}{a^2}\sin 2A+\dfrac{c^2-a^2}{b^2}\sin 2B+\dfrac{a^2-b^2}{c^2}\sin 2C=0.$
3. $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0.$
4. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$
5. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+s+c.$
6. $\dfrac{\sin(B-C)}{\sin(B+C)}=\dfrac{b^2-c^2}{a^2}.$
7. $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}=\dfrac{\sin B}{2a\sin C}$

$$+\frac{\sin C}{2b\sin A}+\frac{\sin A}{2c\sin B}$$

8. $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A.$

9. $a(b \cos C - c \cos B) = b^2 - c^2.$

10. $a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C).$

11. $(a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C.$

12. $$\frac{b^2 - c^2}{a} \cos A + \frac{c^2 - a^2}{b} \cos B + \frac{a^2 - b^2}{c} \cos C = 0.$$

13. $$\frac{\cos B}{\cos C} = \frac{c - b \cos A}{b - c \cos A}.$$

14. $(b+c) \cos A + (c+a) \cos B + (a+b) \cos C = a+b+c.$

15. If the sides of a triangle be 4, 5 and 6, show that the greatest angle is double the least.

16. *If A, B, C be any three points on a line and if P be any point outside the line then prove that*

$$PA^2.BC + PB^2.CA + PC^2.AB = -BC.CA.AB.$$

(Stewarts Theorem).

(The proper sign is attached to the segment on the line)

Sol. Here let $\angle PBA = \theta$

$\therefore \quad \angle PBC = 180° - \theta.$

By cosine formulæ from $\triangle$s PAB and PBC we get

$PA^2 = PB^2 + AB^2 - 2PB.AB \cos \theta \quad \ldots\ldots (i)$

and $PC^2 = PB^2 + BC^2 + 2PB.BC \cos \theta \quad \ldots\ldots (ii)$

Multiplying (i) by BC and (ii) by AB and adding we get

$PA^2.BC + PC^2.AB = PB^2(AB + BC) + AB^2.BC + BC^2.AB$

$PA^2.BC + PC^2.AB = PB^2.AC + AB.BC(AB + BC)$

$\therefore \quad PA^2.BC + PB^2.CA + PC^2.AB = -BC.CA.AB,$

which is the required result.

71. To find the sines of half the angles in terms of the sides.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

But $$\cos A = 1 - 2 \sin^2 \frac{A}{2}.$$

$$\therefore \quad \frac{b^2 + c^2 - a^2}{2bc} = 1 - 2 \sin^2 \frac{A}{2},$$

$$\therefore \quad 2 \sin^2 \frac{A}{2} = 1 - \frac{b^2 + c^2 - a^2}{2bc}.$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)(a-b+c)}{2bc} \quad \dots (i)$$

Now put $2s=a+b+c$

$\therefore \quad a+b-c=a+b+c-2c$

$\qquad\qquad =2s-2c=2(s-c).$

Similarly $\quad a-b+c=2(s-b)$

$\therefore$ From (i), $2\sin^2\frac{A}{2}=\frac{2(s-c)2(s-b)}{2bc}$.

or $\quad \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$

$\therefore \quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$.

Similarly $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$.

and $\quad \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$.

Since A, B, C are angles each less than 180° therefore $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ must be acute. Consequently the above radicals must be taken with a positive sign.

72. **To find the cosines of half the angles in terms of the sides.**

$$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

But $\quad \cos A=2\cos^2\frac{A}{2}-1.$

$\therefore \quad 2\cos^2\frac{A}{2}-1=\frac{b^2+c^2-a^2}{2bc}$

$$\therefore \qquad 2\cos^2\frac{A}{2}=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$=\frac{2s\times 2(s-a)}{2bc}$$

$$\therefore \qquad \cos^2\frac{A}{2}=\frac{s(s-a)}{bc} \text{ or } \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

Similarly $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}$ and $\cos\dfrac{C}{2}=\sqrt{\dfrac{s(s-c)}{ab}}$

The radicals are taken with the positive sign because $\dfrac{A}{2}$, $\dfrac{B}{2}$, $\dfrac{C}{2}$ are all acute.

73. To find the tangents of half the angles in terms of the sides.

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\text{Similarly } \tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

$$\text{and} \qquad \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

The radicals are taken with the positive sign because $\dfrac{A}{2}$, $\dfrac{B}{2}$ and $\dfrac{C}{2}$ are acute.

Another method.

$$\cos A=\frac{b^2+c^2-a^2}{2bc}. \qquad \text{But } \cos A=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}.$$

$$\therefore \quad \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}=\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \quad \tan^2\frac{A}{2}=\frac{2bc-b^2-c^2+a^2}{2bc+b^2+c^2-a^2}$$

$$=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}$$

$$=\frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}$$

$$=\frac{(2s-2c)(2s-2b)}{2s(2s-2a)} \quad [\text{Putting } 2s=a+b+c]$$

$$=\frac{(s-b)(s-c)}{s(s-a)} \quad \therefore \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The radical is taken with the positive sign because $\frac{A}{2}$ is acute.

74. To find the sine of any angle in terms of the sides of a triangle.

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly $\sin B=\frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}.$

and $\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$

Cor. It follows that

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{abc}.$$

Ex. 1. In any triangle ABC show that $c\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)$ $=(a-b)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right).$

Here we are to prove that

$$\frac{a-b}{c}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}} \quad \text{or} \quad \frac{a-b+c}{-a+b+c}=\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}.$$

$$\text{Now } \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}=\sqrt{\frac{(s-c)(s-b)}{s(s-a)}\times\frac{s(s-b)}{(s-a)(s-c)}}=\frac{s-b}{s-a}$$

$$=\frac{a+c-b}{b+c-a}$$

Ex. 2. In any triangle ABC, prove that

$$(b+c-a)\sin\frac{A}{2}=2a\sin\frac{B}{2}\sin\frac{C}{2}$$

$$2a\sin\frac{B}{2}\sin\frac{C}{2}=2a\sqrt{\frac{(s-c)(s-a)}{ca}}\times\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=2(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$=(2s-2a)\sin\frac{A}{2}=(b+c-a)\sin\frac{A}{2}.$$

EXERCISE XXVI

In any $\triangle$ ABC show that :

1. $s=a\cos^2\frac{B}{2}+b\cos^2\frac{A}{2}=b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}$

   $=c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}.$

2. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2.$

3. $\frac{2(a+b)}{c}\sin^2\frac{C}{2}=\cos A+\cos B.$

4. $(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}.$

5. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}.$

6. $2\left(a \sin^2\frac{C}{2}+c \sin^2\frac{A}{2}\right)=c+a-b.$

7. If $\cot\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ be in arithmetical progression, then $\cot\frac{A}{2} \cdot \cot\frac{C}{2}=3.$

8. If $3a=b+c$, prove that $\cot\frac{B}{2}\cot\frac{C}{2}=2.$

9. If in a triangle $c\tan C+b\tan B=(c+b)\tan\frac{C+B}{2}$, then show that $c=b$.

10. In triangles ABC and A'B'C', the angles B and B' are equal and the angles A, A' are supplementary. Show that $aa'=bb'+cc'$.

11. If $\cot A+\cot C=2\cot B$, show that $c^2+a^2=2b^2$.

12. If $3\tan\frac{A}{2}\tan\frac{C}{2}=1$, prove that a, b and c are in arithmetical progression.

13. If the cosines of two of the angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right-angled.

14. If the cosines of two of the angles of a triangle be proportional to the opposite sides, show that the triangle is isosceles.

15. If $\sin(A-B)=2\sin C$, show that $a^2-b^2=2c^2$.

16. The bisector of the angle A of a triangle ABC meets BC in D. Show that

$$AD=\frac{2bc}{b+c}\cos\frac{A}{2}. \qquad \text{(B. U.)}$$

17 In a triangle ABC, right angled at C, prove that

$\tan\frac{A-B}{2}=\frac{a-b}{a+b}$. 18. $\sin^2\frac{B}{2}=\frac{c-a}{2c}$. 19. $\cos^2\frac{A}{2}=\frac{b+c}{2c}$.

Formulae on Chapter X

1. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$. **(Sine Formulæ)**

2. (*i*) $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ R

(*ii*) $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$

(*iii*) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$ (Napier's Analogies).

3. (*i*) $\cos A=\frac{b^2+c^2-a^2}{2bc}$, (*ii*) $\cos B=\frac{c^2+a^2-b^2}{2ca}$

(*iii*) $\cos C=\frac{a^2+b^2-c^2}{2ab}$. (Cosine Formulæ).

4. (*i*) $a=b\cos C+c\cos B$
(*ii*) $b=a\cos C+c\cos A$
(*iii*) $c=a\cos B+b\cos A$. (Projection Formulæ).

5. (*i*) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$.

(*ii*) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$,

(*iii*) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

6. $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

REVISION QUESTIONS VII

1. Given that the sides of a triangle are x^2+x+1, x^2-1, and $2x+1$; find the greatest angle.

2. Show that if the sides of a triangle be in Arithmetical Progression so are the cotangents of its semi-angles.

3. If in a triangle ABC, $\cos B=\frac{\sin C}{2\sin A}$, show that the triangle is isosceles.

4. In any triangle ABC prove that
$$a\sin A-b\sin B=c\sin(A-B).$$

5. In triangle ABC, $a=3$, $b=2\sqrt{3}$ and $A=40°$, find B.

6. Show that if the cosines of two angles of a triangle be directly proportional to the opposite sides, the triangle is

isosceles; but if they are inversely proportional to the opposite sides, then the triangle is either isosceles or right angled.

7. In any triangle ABC, if $A=60°$, then

$$b+c=2a\cos\frac{B-C}{2}.$$

8. In a triangle ABC, $a=3$, $b=5$ and $c=7$. Show that triangle is obtuse-angled and find he obtuse angle.

9. Show that the smallest angle of the triangle whose sides are 10, 17 and 21 is less than 30°.

10. In any triangle ABC, $c=a\cos B+b\cos A$; deduce that $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

11. Show that $c^2=(a+b)^2\sin^2\frac{C}{2}+(a-b)^2\cos^2\frac{C}{2}$.

12. If D be the middle point of the base BC of a triangle ABC, and L and H the points where the bisector of the vertical angle and the perpendicular from the vertex respectively meet the base, prove that DL : DH as $a^2 : (b+c)^2$.

13. In a triangle ABC, show that

$$(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0.$$

14. In any triangle ABC, show that

$$s-c=a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}.$$

15. In a triangla ABC, C is a right angle. Show that

$$a\left(1+\tan\frac{B}{2}\right)=(b+c)\left(1-\tan\frac{B}{2}\right).$$

16. In any triangle ABC, show that

$$\frac{1}{a}\cos^2\frac{A}{2}+\frac{1}{b}\cos^2\frac{B}{2}+\frac{1}{c}\cos^2\frac{C}{2}=\frac{(a+b+c)^2}{4abc}.$$

17. In any triangle ABC, show that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{a+b-c}\cot\frac{C}{2}.$$

18. In any triangle ABC, show that

$$4bc\sin^2\frac{A}{2}+4ca\sin^2\frac{B}{2}+4ab\sin^2\frac{C}{2}$$
$$=2ab+2bc+2ca-a^2-b^2-c^2,$$

19. In any triangle ABC, show that

$$a \sin\frac{B-C}{2} \operatorname{cosec} \frac{A}{2} + b \sin \frac{C-A}{2} \operatorname{cosec} \frac{B}{2}$$
$$+c \sin\frac{A-B}{2} \operatorname{cosec} \frac{C}{2} = 0.$$

Show that the result is still true if all cosecants be changed into secants.

20. If $A+B+C=180°$, show that

$$\tan\frac{A}{2} + \cos\frac{A}{2} \sec\frac{B}{2} \sec\frac{C}{2} = \tan\frac{B}{2} + \cos\frac{B}{2} \sec\frac{C}{2} \sec\frac{A}{2}$$
$$= \tan\frac{C}{2} + \cos\frac{C}{2} \sec\frac{A}{2} \sec\frac{B}{2}.$$

21. $$\frac{a^2 \sin(B-C)}{\sin B+\sin C} + \frac{b^2 \sin(C-A)}{\sin C+\sin A} + \frac{c^2 \sin(A-B)}{\sin A+\sin B} = 0.$$

22. From $a=b \cos C+c \cos B$ and two other similar results' deduce that

$$\cos^2 A+\cos^2 B+\cos^2 C=1-2 \cos A \cos B \cos C.$$

23. The bisector of the angle A of a triangle ABC meets, BC in D. Show that if the square on AD is equal to three quarters of the rectangle contained by the sides AC, AB, then the sides of the triangle are in A. P.

24, If ABC be a triangle then show that

$$\sin 3A \sin(B-C) + \sin 3B \sin(C-A) + \sin 3C \times \sin(A-B) = 0.$$ (B. U.)

25. If $A+B+C+D=2\pi$ prove that

$$\cos(B+C+D) + \cos(C+D+A) + \cos(D+A+B)$$
$$+\cos(A+B+C) + 4 \cos\frac{A+B}{2} \cos\frac{A+C}{2} \times$$
$$\cos\frac{A+D}{2} = 0.$$ (B. U.)

26. If a,b,c be in A. P., prove that $2 \sin\frac{A}{2} \sin\frac{C}{2} = \sin\frac{B}{2}$.

27. The three sides of a triangle are in arithmetical progression, and the greatest angle exceeds the least by a

right angle ; prove that the sides are in the ratios

$$\sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1.$$

28. Prove that in any traingle ABC,

$$\left(b \sin^2\frac{C}{2}+c \sin^2\frac{B}{2}\right) \tan \frac{A}{2}=\left(c \sin^2\frac{A}{2}+a \sin^2\frac{C}{2}\right) \tan \frac{B}{2}$$
$$=\left(a \sin^2\frac{B}{2}+b \sin^2\frac{C}{2}\right) \tan \frac{C}{2}.$$

CHAPTER XI
LOGARITHMS

75. *The logarithm of a certain number N to a base **a** is the index of the power to which the base a must be raised in order to make it equal to the given number.* It follows therefore that if $a^n=N$ then the logarithm of N to base a is n.

Ex. 1. Find the logarithm of 3 to the base 81.

Let x be required number. Then $81^x=3$.

$3^{4x}=3^1$; $\therefore\ 4x=1$ or $x=\frac{1}{4}$.

Ex. 2. Find the logarithm of 128 to the base $\sqrt[3]{4}$.

Let x be the required number. Then $(\sqrt[3]{4})^x=128$:

or $4^{\frac{x}{3}}=2^7$, *i.e.*, $2^{\frac{2x}{3}}=2^7$. $\therefore\ \frac{2x}{3}=7$ or $x=\frac{21}{2}$.

76. The logarithm of N to a given base a is written as $\log_a N$. Hence the two equations $a^x=N$ and $x=\log_a N$ have the same meaning. The student is advised to be quite familiar with this notation and to be able to derive readily one equation from the other.

Important Conclusions—1. Since $a^0=1$, therefore the logarithm of unity to any finite base a is zero ; *i.e.*, $\log_a 1=0$.

2. Since $a^1=a$, therefore the logarithm of the base itself is unity : *i.e.*, $\log_a a=1$.

3. The logarithm of zero to any base other than zero is infinite.

4. The logarithm of a negative number to any positive base is not real.

Examples

1. Since $2^5=32$, $\log_2 32=5$.

2. Since $\frac{1}{81}=\frac{1}{3^4}=3^{-4}$, $\log_3\frac{1}{81}=-4$.

EXERCISE XXVII

Change the following statements from exponential to logarithmic form :

1. $3^5=243$.
2. $2^4=16$.
3. $10^{-2}=0{\cdot}01$.
4. $(16)^{\frac{3}{2}}=64$.

Solve for x :

5. $x=\log_5 25$.
6. $x=\log_{100}10$.
7. $\log_x 4=\frac{2}{3}$.
8. $\log_x 125=3$,
9. Show that $a^{\log_a x}=x$.

[Let $a^{\log_a x}=y$. Then by definition $\log_a y=\log_a x$ $\therefore$ $y=x$.]

10. Show that $\log_a(a^x)=x$.

[Let $\log_a(a^x)=y$. $\therefore$ by definition $a^y=a^x$ $\therefore$ $y=x$.]

77. The student is already familiar with the following laws of indices :

(i) $a^m\times a^n=a^{m+n}$.

(ii) $a^m\div a^n=a^{m-n}$ and (iii) $(a^m)^n=a^{mn}$.

Corresponding to these we have three fundamental laws of logarithms, namley,

(i) $\mathbf{\log_a(mn)=\log_a m+\log_a n}$.

(ii) $\mathbf{\log_a\left(\frac{m}{n}\right)=\log_a m-\log_a n}$

and (iii) $\mathbf{\log_a m^n=n.\log_a m}$.

We shall now prove these laws in the three articles that follow.

78. The logarithm of the product of two factors is equal to the sum of the logarithms of the factors, *i.e.*,

$$\log_a(mn)=\log_a m+\log_a n$$

Let $\log_a m=x$, so that $m=a^x$,

and let $\log_a n=y$, so that $n=a^y$.

$\therefore$ $mn=a^x.a^y=a^{x+y}$.

Hence $\log_a mn=x+y$

$$=\log_a m+\log_a n.$$

Note.—The method of proof is perfectly general and is applicable to any number of factors. Thus $\log_a (mnp\,..)$ $=\log_a m+\log_a n+\log_a p+\ldots$

79. The logarithm of quotient is equal to the logarithm of the numerator diminished by the logarithm of the denominator ; *i.e.*,

$$\log_a\left(\frac{m}{n}\right)=\log_a m-\log_a n.$$

Let $\qquad \log_a m=x$, so that $m=a^x$,
and let $\qquad \log_a n=y$, so that $n=a^y$.

Hence $\dfrac{m}{n}=\dfrac{a}{a^y}=a^{x-y}$.

$\therefore$ by definition, $\log_a\left(\dfrac{m}{n}\right)=x-y=\log_a m-\log_a n.$

80. The logarithm of any power of a number is equal to the product of the index of the power and the logarithm of the number ; *i.e.*,

$$\log_a m^n=n\log_a m.$$

Suppose $\log_a m=x$; $\quad\therefore\ a^x=m.$
Hence $\qquad m^n=(a^x)^n=a^{nx}$;
$\therefore \qquad \log_a m^n=nx=n\log_a m.$

Ex. 1. Show that

$$\log\left(\frac{a^x\times b^y}{l^m\times p^r}\right)=x\log a+y\log b-m\log l-r\log p,$$

w. r. t. any base.

$$\log\left(\frac{a^x\times b^y}{l^m\times p^r}\right)=\log(a^x+b^y)-\log\,(l^m\times p^r)$$
$$=\log a^x-\log b^y-(\log l^m+\log p^r)$$
$$=x\log a+y\log b-m\log l-r\log p.$$

Caution :—The student is advised to note carefully that $\log_a(m+n)$ *is not equal to* $\log_a m+\log_a n$. *In fact there is no formula for* $\log_a(m+n)$ *connecting it with* $\log_a m+\log_a n$.

81. *Natural Logarithms.* When the base used is e which stands for the infinite converging series :

$1+1+\dfrac{1}{\underline{|2}}+\dfrac{1}{\underline{|3}}$ .... the logarithms are called Natural Logarithms. It can be easily proved that e lies between 2

and 3. This system is used only in Higher Mathematical investigations and is not suitable for numerical calculations.

Common Logarithms or Brigg's System. When the base used is 10, the logarithms are called Common Logarithms.

This system has got several advantages as we shall shortly see.

We shall at present restrict ourselves to the study of the common logarithms.

82. The logarithm of a number is not always integral. Thus since $10^2=100$ and $10^3=1000$, the logarithm of a number lying between 100 and 1000 lies between 2 and 3 and is therefore equal to 2+ a positive proper fraction. Similarly since ·00845 lies between ·001 and ·01, *i. e.*, between 10^{-3} and 10^{-2}, the logarithm of ·00845 is greater than −3 and less than −2, *i. e.*, it is equal −3+a *positive* proper fraction. Whenever a logarithm consists partly of an integer (positive or negative) and partly of a *positive* proper fraction, the integral portion is called the **characteristic** and the *positive* fractional portion is called the **mantissa**. Thus, 5·234 be the logarithm of a certain number, then 5 is the characteristic and ·234 the mantissa; if −4+·1095 be the logarithm of a certain number, then −4 is the characteristic and ·1095 the mantissa. Note that −4 +·1095 = −3·8905. But −3 is not the characteristic, nor −·8905 is the mantissa. **A fractional portion, in order to be called a mantissa, must be positive and only then the integral portion can be called the characteristic.** If the fractional portion is not positive, make it so before calling it a mantissa.

Ex. The logarithm of a number is −8·236. Find the characteristic and the mantissa.

$$-8·236=-8-·236=-9+1-·236=-9+·764.$$

∴ Characteristic is −9 and mantissa is ·764.

Notation.—For the sake of brevity −9+·764 is written as $\bar{9}$·764. The student should note that in $\bar{9}$·764, 9 alone is negative, while ·764 is positive, but in −9·764, both 9 and ·764 are negative. $\bar{9}$ is read as nine bar.

83. *Advantages of the Common System.* **The common system of logarithms possesses the following two very important advantages.**

(1) The characteristic of the logarithm of any number can always be found by inspection.

(2) The mantissæ of the logarithms of all numbers consisting of the same digits arranged in the same order (*i. e.*, of numbers, which differ from each other only in the position of the decimal point) are always the same.

It is now proposed to prove these two statements in next two articles.

84. *To show that the characteristic of the logarithm of any number N can be written down by inspection.*

(*i*) Let the number N be greater than unity having n digits in its integral part.

Then since $10^{\circ}=1$,

$10^{1}=10$,

$10^{2}=100$,

$10^{3}=1000$, and so on,

it follows that a number having one digit in its integral part lies between 10° and $10'$; a number having two digits in its integral part lies between $10'$ and 10^{2}; a number having 3 digits lies between 10^{2} and 10^{3}; and so on. Hence the given nnmber N, having n digits in its integral part, lies weteen 10^{n-1} and 10^{n}.

Hence $N=10^{n-1+k}$ where k is a positive proper fraction.

$\therefore \quad \log N=(n-1)+k$.

Hence the characteristic is $n-1$.

Therefore the characteristic of the logarithm of any number greater than unity is one less than the number of digits in the integral part of the number.

(*ii*) Let the number M be positive and less than unity: also when converted to decimal form, let N have n cyphers immediately after the decimal point.

Since $10^{\circ}=1$.

$10^{-1}=\cdot 1$.

$10^{-2}=\cdot 01$.

$10^{-3}=\cdot 001$, and so on.

it follows that a decimal fraction having no cypher immediately after the decimal point being greater than $\cdot 1$ and less

than 1, lies between 10^{-1} and 10^{0}; a number having one cypher immediately after the decimal point being greater than ·01 and less than ·1, lies between 10^{-2} and 10^{-1}; a number having two cyphers immediately after the decimal point being greater than ·001 and less than ·01, lies between 10^{-2} and 10^{-3} and so on. Hence the given number N, having n cyphers immediately after the decimal point, lies between $10^{-(n+1)}$ and 10^{-n}.

Hence $N=10^{-(n+1)+k}$, where k is a positive proper fraction.

Therefore $\log N=-(n+1)+k$.

Hence the characteristic is $-(n+1)$.

Therefore the characteristic of the logarithm of a decimal fraction is negative and numerically greater by one than the number of cyphers immediately after the decimal point.

Thus the characteristics of the logarithms of the numbers 5678, 56·72 and 587·2 are respectively 3, 1, and 2; and the characteristics of the logarithms of the numbers ·0025, ·02506, and ·50208, are -3, -2 and $-\mathbf{1}$ respectively.

85. *The mantissae of the logarithms of all numbers consisting of the same digits arranged in the same order (i.e., of numbers which differ from each other only in the position of the decimal point) are always the same.*

Let N be a given number and let i be the characteristic and f the mantissa of its logarithm, so that the logarithm of N is $i+f$.

Now in order to obtain a number which differs from N only in the position of the decimal point and consequently has the same digits arranged in the same order, we multiply N by 10^{p} where p is an integer, positive or negative.

$$\text{But } \log (N\times 10^{p})=\log N+\log 10^{p}$$
$$=i+f+p.$$

Hence since i and p are both integers and consequently $i+p$ is an integer, the mantissa f has not changed; it is the same for N as well as for $N\times 10^{p}$.

Ex. 1. Given that $\log 2=·3010$, find the number of digits in 2^{76} and the position of the first significant figure in 2^{-35}.

We have $\log 2^{76} = 76 \log 2 = 76 \times \cdot 3010$
$$= 22\cdot 8760.$$

Since the characteristic of the logarithm of 2^{73} is 22, it follows that in 2^{76} there are 23 digits.

Again, $\log 2^{-35} = -35 \log 2 = -35 \times \cdot 3010 = -10\cdot 5350$
$= \overline{11}\cdot 4650.$

Since the characteristic of the logarithm of 2^{-35} is -11, it follows that there are 10 cyphers following the decimal point, *i.e.*, the first significant figure is in the eleventh place of decimals.

Ex. 2. Given that $\log 3 = \cdot 4771$, $\log 7 = \cdot 8451$ and $\log 11 = 1\cdot 0414$, solve the equation
$$3^2 \times 7^{2x+1} = 11^{x+5}.$$

Taking logarithms, we have
$$\log 3^x + \log 7^{2x+1} = \log 11^{x+5}.$$

$\therefore \quad x \log 3 + (2x+1) \log 7 = (x+5) \log 11.$

$\therefore \quad x (\log 3 + 2 \log 7 - \log 11) = 5 \log 11 - \log 7,$

$$\text{or} \quad x = \frac{5 \log 11 - \log 7}{\log 3 + 2 \log 7 - \log 11}$$
$$= \frac{5\cdot 2070 - \cdot 8451}{\cdot 4771 + 1\cdot 6902 - 1\cdot 0414}$$
$$= \frac{4\cdot 3619}{1\ 1259} = 3\cdot 8 \text{ nearly.}$$

EXERCISE XXVIII

1. Find the values of :
   (*i*) $\log_4 256$. (*ii*) $\log_8 16$. (*iii*) $\log_{81} 243$.

2. Given that $\log 2 = \cdot 3010$, find the values cf
   (*i*) $\log \cdot 0005$. (*ii*) $\log (6\cdot 4)^{-3}$.

Given that $\log 2 = \cdot 3010$, $\log 3 = \cdot 4771$, $\log 7 = \cdot 8451$, find the following logarithms:

3. $\log 14$. 4. $\log 49$. 5. $\log 98$. 6. $\log \sqrt{6}$.
7. $\log 300$.

8. Given that $\log 3 = \cdot 4771$, find the number of digits in
   (*i*) 3^{43}. (*ii*) 3^{27}. (*iii*) 3^{62}.

9. Find the position of the first significant figure in
   (*i*) 3^{-13}. (*ii*) 3^{-43} and (*iii*) 3^{-61}.

10. Solve the equation

$5^{7-x^4}=2^{x+5}$, given that $\log 2=\cdot 3010$.

11. Given that $\log 2-\cdot 3010$, $\log 3-\cdot 4771$ and $\log 7=\cdot 8451$, solve the equations

(*i*) $2^{2x+1}\times 3^{2+2x}=7^{4x}$

(*ii*) $7^{x+y}\times 3^{2x+y}-9$; $3^{x+y}=3^x\times 2^{x-y}$.

12. given $\log 2=\cdot 3010$ and $\log.3=\cdot 4771$, find the logarithm of 12 to the base 40.

13. Show that $\left(\frac{81}{80}\right)^{1000}$ is greater than 100,000.

14. Show that $3^{\frac{1}{3}}>2^{\frac{1}{2}}$.

15. Prove that $\log_a \dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=2\log_a [x+\sqrt{x^2-1}]$.

[Hint : Rationalise the fraction]

86. *Tabular Logarithms.*

As the sine and cosine of an angle are never greater than unity, the characteristics of their logarithms are negative; and the same is true for the tangent of an angle less than 45° or the cotangent of an angle greater than 45° and less than 90°. In such cases the introduction of negative characteristics is avoided by using another system of logarithms called *tabular logarithms* defined thus :

The tabular logarithm of any trigonemetric function is the common logarithm of that function increased by ten. For the sake of brevity, tabular logarithm is denoted by L instead of log.

Thus $L\ \sin\theta=10+\log\ \sin\theta$.

$L\ \tan\theta=10+\log\tan\theta$, and so on.

Ex. Given $\log 2=\cdot 3010$ and $\log 3=\cdot 4771$, find

L sin 45°, L. tan 30°, L cosec 60°.

$$\sin 45^\circ=\frac{1}{\sqrt{2}}=(2)^{-\frac{1}{2}}$$

$$\therefore\ \log\sin 45^\circ=-\frac{1}{2}\log 2=-\cdot 1505=\bar{1}\cdot 8495.$$

Hence $L\sin 45^\circ=10+\log\sin 45^\circ=9\cdot 8495$.

$$\tan 30^\circ=\frac{1}{\sqrt{3}}=(3)^{-\frac{1}{2}}$$

$$\therefore\quad \log\tan 30^\circ=-\tfrac{1}{2}\log 3=-\cdot 2382=\bar{1}\cdot 7618,$$

and $\therefore$ L tan $30° = 10 + \log \tan 30° = 9{\cdot}7618$

$$\text{cosec } 60° = \frac{2}{\sqrt{3}} = 2 \times (3)^{-\frac{1}{2}}$$

$\therefore \log \text{cosec } 60° = \log 2 - \frac{1}{2} \log 3$
$= {\cdot}3010 - {\cdot}2386 = {\cdot}0624,$

and $\therefore$ log cosec $60° = 10 + \log \text{cosec } 60° = 10{\cdot}0624.$

87. To show that $\mathbf{\log_a m = \log_b m \times \log_a b}$.

Let $\log_a m = x$, so that $a^x = m$;
also let $\log_b m = y$, so that $b^y = m$,

$\therefore \quad a^x = b^y$

Hence $\log_a(a^x) = \log_a(b^y)$
But $\log_a(a^x) = x \log_a a = x.$
and $\log_a(b^y) = y \log_a b$;
$x = y \log_a b.$

Hence $\log_a m = \log_b m \times \log_a b$

Cor. 1. $\log_b m = \dfrac{\log_a m}{\log_a b}.$

This formula is used when it is required to transform logarithms from one base to another.

Cor. 2. $\log_b a = \dfrac{1}{\log_a b}$ (putting $m = a$ in Cor. 1).

Also thus :—Let $\log_b a = x$, so that $b^x = a$.

Now, since $a = b^x$, therefore raising both the sides to the power $\dfrac{1}{x}$, we have $a^{\frac{1}{x}} = b.$

Hence by definition $\log_a b = \dfrac{1}{x} = \dfrac{1}{\log_b a}.$

Note. It follows that $\log_a b \times \log_b a = 1$.

Ex. 1. Find the value of $\log_2 3$, given that $\log 2 = {\cdot}3010$ and $\log 3 = {\cdot}4771$

$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = \frac{{\cdot}4771}{{\cdot}3010} = 1{\cdot}5850.$$

Ex. 2. Evaluate $\log_2 10$ to two decimal places given that $\log_{10} 2 = {\cdot}3010.$

$$\log_2 10 = \frac{1}{\log_{10} 2} = \frac{1}{{\cdot}3010} = 3{\cdot}32.$$

EXERCISE

1. **Prove that** $\log_a b \times \log_b a = 1$.

Evaluate the following logarithms to two places of decimals ;

2. $\log_2 3$.   3. $\log_3 2$.   4. $\log_3 10$.
5. $\log_{27} 64$.   6. $\log_{.30} 2$.

Ans. 1·58, ·63, 2·09, 126, $\bar{1}$·81.

88. How to Use the Four Figure Log Tables ;

(1) To find the logarithm of a given number.

Note.—Only the mantissae are given in these tables, the characteristic in each case being found by two well-known rules, given before.

Mantissae of logs of all numbers form 1 to 9999, i.e., of numbers consisting of four *significant* digits can be found. The following directions indicate the method of using such a table :

(*i*) The extreme left-hand column, at the top of which there is a vacant square, corresponds to the first two significant figures of the numbers.

(*ii*) The next ten columns are headed 0, 1, 2...9; correspond to the third figure of the given number.

(*iii*) The small columns to the extreme right (generally called "difference columns") are similarly headed 1, 2...9 : and these figures correspond to the fourth significant figure in the given number.

The method of using the tables is illustrated in the following example.

Find the logarithm of 4597.

In the first column look for 45 (first two figures in the given number, in the same horizontal line as 45 and in the column under number 9 (the third figure in the given number) we get the number 6618 : under 7 (the fourth figure in the given number) in the small difference column and in the same row as 45 we find 7. This means that 6618 and 7 are to be added : their sum being 6625, the mantissæ in the log of 4597 is ·6625, and the characteristic (not given in the tables) is evidently 3.

Hence $\log 4597 = 3·6625$.
Similarly $\log 45·97 = 1·6625$
and $\log ·04597 = \bar{2}·6625$.

(2) To find the number whose logarithm is given.

Tables of anti-logarithms are used in this case and they are used exactly in the same way as logarithm tables explained before.

Ex. Find the number whose logarithm is 2·9072.

Let x be the number, ∴ log x=2·9072.

To find x we leave the characteristic 2 for the present and take the mantissa ·9072 only.

Turn to anti-log tables : run down the first *column* till ·90 (the first two figures in the given log) is reached ; then in the *horizontal row* containing ·90 and under the column headed by 7 (the third figure) is the number 8072 ; and in the difference column headed by 2 (the fourth figure) and in the same *horizontal row* as ·90 is found the number 4. This 4 is added to 8072 and the sum 8076 is the number corresponding to the mantissa 2·9072. Now since the given characteristic is 2, therefore x shall contain three figures in its integral part and hence combining the two facts, x=807·6.

Similarly the number whose log is 1·9072 is 80·76 and the number whose log is $\bar{2}$·9072 is ·08076.

(3) To find the trigonometrical functions of an angle from their tables. This has already been explained before. (*See* Chapter III).

(4) To find the angles corresponding to a given trigonometric function from their tables. This has already been explained before (*See* Chapter III).

(5) To find the logarithmic trigonometric ratio of a given angle.

The tables give (*i*) the logarithmic trigonometric functions of all angles from 0° to 90° at intervals of 6′ : (*ii*) and also contain *difference columns* of angles 1, 2, 3, 4, 5 minutes.

The method of using the tables is illustrated by the following example.

Ex. 1. Find log sin 49° 36′

(*a*) The first column in log sine page contains degrees look for the *row* containing 49° ;

(*b*) Look for the *column* which is headed 36′ ;

(*c*) At the *point of the intersection of the row and the*

column we get the number 8817, which is the mantissa of log sin 49° 36′, the characteristic $\bar{1}$ being shown only in the column under 0′.

Note.—The number shown there is 9, but according to the tabular logarithms it is 10 more than the required result, *i.e.*, characteristic is 9—10=—1).

$\therefore$ log sin 49° 36′ = $\bar{1}$·8817.

Ex. 2. Find log sin 48° 35′.

Here 35′ is not found at the top of any of the columns. the *mean difference columns* are, therefore, to be used.

(*a*) Take the *row* containing 48°.

(*b*) Take the column headed 30′ (which is less than 35′ and which is found at the top of a certain column) and we get at their point of intersection 8745, which is the mantissa of log sin 48° 30′. Now we have to find the difference for 5′.

(*c*) Look for the number *in the same row* as 48° in the difference column under 5 and we get the number 6 there. This 6 is to be added to 8745 obtained above, thus the sum is 8751 and $\therefore$ log sin 48° 35′ = $\bar{1}$·8751

Note.—It appears from the tables that log sin 48° 35′ and log sin 48° 36′ are the same which is evidently absurd. The inference is that they are equal up to 4 places of decimals only—there *must* be some difference somewhere after the fourth place of decimals.

(6) To find the angle corresponding to a given logarithmic trigonometric function *e.g.*,

given log sin x = $\bar{1}$·9182 ; find x.

The given number $\bar{1}$·9182 cannot anywhere be found in the table ; but we get 9181 which is nearest to 9182 and less than it. We get 9181 under 54′ and in the row of 55°. This shows that log sin 55° 54′ is $\bar{1}$·9181, Now the difference between 9182, the given mentissa, and 9181 is 1, and this difference 1 found in the *difference* columns under 1, and this 1′ is to be added to 55° 54′.

i.e., $x = 55° 55′$.

Note.—As the angle θ increases from 0° to 90°, the difference is additive in the case of log sin θ and log tan θ and subtractive in the case of log cos θ and log cot θ.

Ex. 1. Find the value of $\dfrac{(435)^3 \sqrt{\cdot 056}}{(380)^4}$ as accurately as you can with the help of four figure log tables.

Sol. Let $x=\dfrac{(435)^3\sqrt{\cdot 056}}{(380)^4}$.

$\therefore$ $\log x = 3\log 435 + \frac{1}{2}\log\cdot 056 - 4\log 380$
$= 3\times 2\cdot 6385 + \frac{1}{2}\times \bar{2}\cdot 7482 - 4\times 2\cdot 5798$
$= 7\cdot 9155 - 1 + \cdot 3741 - 10\cdot 3192$
$= -3\cdot 0296 = -4 + 1 - \cdot 0296$
$= \bar{4}\cdot 9704$ $\therefore$ $x = \cdot 0009342$.

Ex. 2. Find the value of $\dfrac{(3\cdot 142)^3\times(\cdot 078)^{\frac{1}{3}}}{(\cdot 005)^{\frac{1}{4}}}$

as accurately as you can.

Sol. Let $x=\dfrac{(3\cdot 142)^3\times(\cdot 078)^{\frac{1}{3}}}{(\cdot 005)^{\frac{1}{4}}}$

$\therefore$ $\log x = 3\log 3\cdot 142 + \frac{1}{3}\log\cdot 078 - \frac{1}{4}\log\cdot 005$
$= 3\times 0\cdot 4972 + \frac{1}{3}\times \bar{2}\cdot 8921 - \frac{1}{4}\times \bar{3}\cdot 6990$
$= 1\cdot 4916 + \frac{1}{3}(-3 + 1\cdot 8921) - \frac{1}{4}(-4 + 1\cdot 6990)$
$= 1\cdot 4916 - 1 + \cdot 6307 + 1 - \cdot 42475$
$= 1\cdot 69755$
$= 1\cdot 6976$, correct to 4 places of decimals.
$x = 49\cdot 84$.

Ex. 3. The period T of small oscillation of a simple pendulum of length l is given by $T=2\pi\sqrt{\dfrac{l}{g}}$. Calculate the value of g when it is observed that the period of oscillation of a pendulum 46·2 cm. long is 1·36 sec.

Taking logarithms, we have

$\log T = \log 2 + \log\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$
$\therefore$ $\log g = 2\log 2 + 2\log 22 - 2\log 7 + \log 46\cdot 2 - 2\log 1\cdot 36$
$= \cdot 6020 + 2\cdot 6848 - 1\cdot 6902 + 1\cdot 6646$
$= 2\cdot 9242$
$\therefore$ $g = 986\cdot 8$.

EXERCISE XXIX

With the aid of four figure log tables, find the value of:

1. $\dfrac{3\cdot 274\times\cdot 0059}{14\cdot 83\times\cdot 0077}$. 2. $\dfrac{15\cdot 38\times\cdot 0137}{\cdot 276\times\cdot 0038}$. 3. $\sqrt{\dfrac{\cdot 0137\times\cdot 0296}{873\cdot 5}}$.

4. The mean propotional between 2 87 and 30·08.
5. The third proportional to ·0238 and 7·805
6. The mean proportional between $\sqrt[3]{\cdot 3473}$ and $\sqrt[5]{\cdot 256.4}$

7. Find log $\tan\frac{A}{2}$ (given log 2=0·3010300 and log 3= 0.4771213) when $a=18$, $b=20$, $c=22$. (C. U.)

8. If Rs. P are invested at $r\%$ compound interest it amounts after n years to Rs. A where $A=P\left(1+\frac{r}{100}\right)^n$. Find A if $P=250$, $r=4$ and $n=12$.

9. Simplify : $(\cdot 008)^{-\frac{1}{3}}+(\cdot 35)^{-\frac{1}{2}}$. [Hint. Simplify each term separately].

10. Find x from the equation $4^{\frac{1}{x}}=9$.

11. Solve for x : $4^{2x}-8\times 4^x+12=0$.

89, *The principle of Proportional Parts.*

If we are required to find the logarithm of a given number whi·h is not continued in the tables or the number corresponding to a given logarithm not given in the tables, we apply the Principle of Proportional Parts which states that *the increase in the logarithm of a number is proportional to the increase in the number itself.*

Similarly if we are required to find the trigonometrical ratio of an angle which is not contained in the tables, or the angle corresponding to a given trigonometrical ratio, we apply the fact that *the change in the trigonometric ratio of an angle is proportional to a small change in the angle itself.*

Note.—The changes referred to above must be small otherwise errors are bound to appear,

For example, let it be given that log 2=·3010 and log 3 =·4771 and let us find log ·25 from the above principle.

log 3=·4771
log 2=·3010.

For a change of a unity in the number, the change in the logarithm is ·1761; therefore for the change ·5 in the number the change in logarithm must be ·08805.

Therefore log 2·5=·3010+·08805=·38905.

But from tables log $2{\cdot}5 = .3979$. Thus the result is correct only up to the first place of decimals.

Ex. 1. Given that $\log 93{\cdot}15 = 1{\cdot}9691$,
and $\log 93{\cdot}16 = 1{\cdot}9692$, find $\log 93{\cdot}155$.
Since $\log 93{\cdot}16 = 1{\cdot}9692$
and $\log 93{\cdot}15 = 1{\cdot}9691$.
$\therefore$ difference in the log for $\cdot 01 = \cdot 0001$.

$\therefore$ difference in the log for $\cdot 005 = \dfrac{\cdot 0001 \times \cdot 005}{\cdot 01}$

$= \cdot 00005$.

$\log 93{\cdot}155 = 1{\cdot}9691 + \cdot 00005$
$= 1{\cdot}96915$.

Ex. 2. Given that $\log \sin 21°3' = \bar{1}{\cdot}5553$,
and $\log \sin 21°\,4' = \bar{1}{\cdot}5556$,
find $\log \sin 21°3'\,20''$
Since $\log \sin 21°\,4' = \bar{1}{\cdot}5556$,
and $\log \sin 21°\,3' = 1{\cdot}5553$.
$\therefore$ difference for $60'' = \cdot 0003$
$\therefore$ proportional difference for $20'' = \cdot 0003 \times \frac{1}{3} = \cdot 0001$
$\therefore$ $\log \sin 21°\,3'\,20'' = \bar{1}{\cdot}5553 + \cdot 0001$
$= 1{\cdot}5554$.

Note.—It may be remarked, however, that with four-figure table we can only aim at finding angles correct to the nearest minute ; in many cases we cannot do even this with certainty; therefore while using four-figure tables, we shall seldom use this principle.

Solved Examples

Ex. 1. In the triangle ABC, $c=70$ ft., $a=42$ ft. $C=90°$, find A, B and b.

$$\sin A = \frac{a}{c}$$

$\therefore$ $\log \sin A = \log a - \log c = \log 42 - \log 70$
$= 1{\cdot}6232 - 1{\cdot}8451 = -\cdot 2219$
$= \bar{1}{\cdot}7781$.

$\therefore$ $A = 36°52'$. $\therefore$ $B = 90° - A = 53°\,8'$.

Again $\dfrac{b}{c} = \sin B$, or $b = c \sin B$

$\therefore \log b = \log c + \log \sin B = \log 70 + \log \sin 53°8'$
$= 1\cdot8451 + \bar{1}\cdot9031$
$= 1\cdot7482.$
$b = 56\cdot01$ ft.

Ex. 2. In the triangle ABC, $b=7\cdot771$, $B=51°$, $C=90°$ find A, a and c.

$$A = 90° - B = 90° - 51° = 39°$$

$$\frac{a}{b} = \tan A, \text{ or } a = b \tan A.$$

$$\therefore \quad \log a = \log b + \log \tan A$$
$$= \log 7\cdot771 + \log \tan 39°$$
$$= \cdot8905 + \bar{1}\cdot9084 = \cdot7989$$
$$\therefore \quad a = 6\cdot294.$$

Again $$\frac{b}{c} = \sin B$$

$$\therefore \quad c = \frac{b}{\sin B}.$$

$$\therefore \quad \log c = \log b - \log \sin B$$
$$= \log 7\cdot771 - \log \sin 51° = \cdot8905 - \bar{1}\cdot8905 = 1.$$
$$c = 10.$$

Ex. 3. Given that $\frac{a}{\sin A} = \frac{b}{\sin B}$; find a if

$$b = 93\cdot24,\ A = 53°31',\ B = 49°\ 36'.$$

Since $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

$$\therefore \quad a = \frac{b \sin A}{\sin B}.$$

$$\therefore \quad \log a = \log b + \log \sin A - \log \sin B$$
$$= 1\cdot9696 + \bar{1}\cdot9052 - \bar{1}\cdot8817$$
$$= 1\cdot9931$$
$$\therefore \quad a = 98\cdot42.$$

EXERCISE XXX

1. Solve $3^x = 2$ to three places of decimals. (C. U. 1927).

2. In the triangle ABC, $C=90°$, $c=32\cdot3$ and $a=16\cdot7$, find A.

3. In the triangle ABC, C=90°, A=32° 13′, a=16·83, find b and c.

4. In the triangle ABC, C=90°, c=29·9 and B=56°38′, find b, a and A.

5. Given that $\frac{a}{\sin A}=\frac{b}{\sin B}$, find a when b=33·2, A=43° 31′ and B=52° 20′.

6. Given that $\frac{a}{\sin A}=\frac{b}{\sin B}$, find B when a=98·42, b=93·24 and A=53° 31′.

7. Given that $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, find A when $a=4$, $b=8$, $c=11$.

REVISION QUESTIONS VIII

1. Define logarithm. Prove that

$x^n=a^{n\log_a x}$. Hence show that $x^3=e^{3\log_e x}$.

Find the lagarithm of 3 to the base 2.

2. State and prove the advantages of the common system of logarithms.

With the help of tables, find the geometric mean between $(·03569)^{\frac{2}{5}}$ and $(2.879)^{\frac{3}{7}}$.

3. Find the logarithm of the product or the quotient of two numbers in terms of the logarithms of those numbers.

Also prove that $\log m^n=n\log m$.

If, x, y, z be in H. P., prove that

$$\log(x+z)+\log(x-2y+z)=2\log(x-z).$$

4. Distinguish between characteristic and mantissa of a logarithm.

Having given log 2=·30103, find the number of digits in 2^{37} and the position of the first significant figure in 2^{-37}.

a=345·6 and b=283·5. Find the value of $\sqrt{a^2-b^2}$.

5. If $a^2+b^2=7ab$, prove that

$$\log[\tfrac{1}{3}(a+b)]=\tfrac{1}{2}(\log a+\log b).$$

Find the value of

(*i*) $\sqrt{\dfrac{1728\times 1{\cdot}21}{245}}$. (*ii*) $\dfrac{({\cdot}439)^{\frac{2}{3}}\times(14{\cdot}7)^3}{({\cdot}0062)^{\frac{1}{4}}}$.

6. Write down the values of

(*i*) tan 62° 31′. (*ii*) sin 59° 21′. (*iii*) cos 78° 43′.

(*iv*) log sin 80° 18′. (*v*) log cos 84° 36′. (*vi*) log tan 9° 19′.

7. Prove that

$$\log_a m = \log_b m \times \log_a b.$$

Show how to convert logarithms of numbers from the natural to the common system or from the common to the natural system.

8. If a, b, c be in G. P., show that

$\log_a x$, $\log_b x$, $\log_c x$ are in H. P.

9. Prove that

(*i*) $\text{Anti-log}_a{}^x \times \text{anti-log}_a{}^y = \text{anti-log}_a{}^{x+y}$,

(*ii*) $\text{Anti-log}_a{}^{xy} = (\text{anti-log}_a{}^x)^y = (\text{anti-log}_a{}^x)$.

10. The post-office 5 years cash certificates for Rs. 500 are obtainable at an issue price of Rs. 440 as. 10. Find the rate per cent.

[Hint : use the Formula :—

$$\text{Amount} = \text{Principal}\left(1+\frac{\text{Rate}}{100}\right)^n.$$ (P. U. 1940)

11. Solve : $\log(x-9)^2+\log(x-4)^2=2$. (B. U.)

12. If P, the centrifugal force on a rotating body is $\dfrac{wv^2}{gr}$, find the value of P when $w=28$, $v=4{\cdot}65$, $g=32{\cdot}2$ and $r=1{\cdot}88$.

13. In the formula $N=30\pi\sqrt{\dfrac{12\,g\,E\,I}{wl^4}}$, find N if $\pi=3{\cdot}142$, $g=32{\cdot}2$, $E=180\times16^6$, $w=0{\cdot}28$ $l=48$ and $I=0{\cdot}0564$.

CHAPTER XII

THE SOLUTION OF TRIANGLES

90. The three angles and the three sides of a triangle are called the six elements of the triangle. When any three elements of a triangle are given at least one of them being a side, the triangle is in general completely known. The process of finding the unknown elements from known ones is called the Solution of the Triangle.

The student is already familiar with the methods for solving the triangle when it is right-angled. We shall now discuss the case of an oblique-angled triangle. The different cases to be considered are:

Case 1. The three sides given.
Case 2. Two sides and the included angle given.
Case 3. One side and two angles given.
Case 4. Two sides and the angle opposite to one of them given.

It may be observed that in every case of a solution of a triangle the best check on the results obtained by calculation can be done by drawing the figure to scale and measuring the required elements.

91. Case 1. *To solve the triangle when the three sides are given.*

Let the three sides a, b, c, of the triangle ABC be given. Then s, $s-a$, $s-b$ and $s-c$ can be found.

Also $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$\therefore \log\tan\frac{A}{2}=\tfrac{1}{2}\left[\log(s-b)+\log(s-c)-\log s-\log(s-a)\right]$$

Whence $\frac{A}{2}$ can be obtained with the help of the tables. Doubling the result we get A. Similarly B can be found from the formula for $\tan\frac{B}{2}$ and then C is known from the equiton $C=180^\circ-(A+B)$.

Note 1.—The expression for $\sin\frac{A}{2}$, $\sin\frac{B}{2}$ and $\sin\frac{C}{2}$ can also be used to find A B and C; but tangent formulæ are the most convenient as the logarithms used in finding $\log\tan\frac{A}{2}$ are the same as those required for finding $\log\tan\frac{B}{2}$ and $\log\tan\frac{C}{2}$.

Note 2.—If the numbers involved are small then the equation $\cos A=\frac{b^2+c^2-a^2}{2bc}$ and two such others can also be used to find the angles

Ex. Given $a=31{\cdot}9$, $b=56{\cdot}31$ and $c=40{\cdot}27$; find the angles of the triangle ABC.

$$s=\frac{31{\cdot}9+56{\cdot}31+40{\cdot}27}{2}=64{\cdot}24$$

$$\therefore\quad s-a=64{\cdot}24-31{\cdot}9=32{\cdot}34$$
$$s-b=64{\cdot}24-56{\cdot}31=7{\cdot}93$$
$$s-c=64{\cdot}24-40{\cdot}27=23{\cdot}97.$$

Now $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{7{\cdot}93\times23{\cdot}97}{64{\cdot}24\times32{\cdot}34}}$.

$$\therefore\quad \log\tan\frac{A}{2}=\tfrac{1}{2}(\log 7{\cdot}93+\log 23{\cdot}97-\log 64{\cdot}24-\log 32{\cdot}34)$$
$$=\tfrac{1}{2}[0{\cdot}8993+1{\cdot}3797-1{\cdot}8078-1{\cdot}5097]$$
$$=\tfrac{1}{2}(-1{\cdot}0385)=-{\cdot}51925=\bar{1}{\cdot}48075$$
$$=\bar{1}{\cdot}4808,$$

correct to 4 places of decimals.

$$\therefore\quad \frac{A}{2}=16^\circ\ 50' \text{ and } \therefore\quad A=33^\circ\ 40'.$$

Again, $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{32{\cdot}34\times23{\cdot}97}{64{\cdot}24\times7{\cdot}93}}$

$$\therefore\quad \log\tan\frac{B}{2}=\tfrac{1}{2}[\log 32{\cdot}34+\log 23{\cdot}97-\log 64{\cdot}24-\log 7{\cdot}93]$$
$$=\tfrac{1}{2}[1{\cdot}5097+1{\cdot}3797-1{\cdot}8078-0{\cdot}8993]$$
$$=\tfrac{1}{2}\times{\cdot}1823={\cdot}09115={\cdot}0912,$$ correct to 4 places of decimals.

$$\therefore\quad \frac{B}{2}=50^\circ\ 58'\ 24'' \text{ and } \therefore\quad B=101^\circ\ 56'\ 48''.$$

Also $C=180^\circ-A-B=44^\circ\ 23'\ 12''$.

But if the Principle of Proportional Parts is not used,

then $\dfrac{B}{2}=50^\circ\ 58'$ and $\therefore\ B=101^\circ\ 56'$

and $\therefore\ C=44^\circ\ 24'$.

EXERCISE XXXI

1. Find the greatest angle of the triangle whose sides are 75·2, 86·4, 94·8 ft.

2. Find the greatest angle of a triangle whose sides are 6, 7 and 8 inches.

3. The sides of a triangle are 32, 40, 66. Find the greatest angle.

4. Find the smallest angle of the triangle whose sides are 18·1, 18·9, 18·5.

5. Solve the triangle, given $a=31{\cdot}9$, $b=56{\cdot}31$, and $c=40{\cdot}27$.

Solve the following triangles and check the solutions :

6. $a=17{\cdot}6$; $b=60{\cdot}24$; $c=33{\cdot}4$.
7. $a:b:c=5:7:8$. 8. $a=1{\cdot}3$; $b=1{\cdot}4$; $c=1{\cdot}5$.
9. $a=428$; $b=283$; $c=317$.
10. $a=87{\cdot}6$; $b=57{\cdot}4$; $c=46{\cdot}8$.
11. $a=58{\cdot}73$; $b=49{\cdot}24$; $c=52{\cdot}31$.
12. $a=15$; $b=22$; $c=9$.
13. If $a=32$, $b=40$, $c=66$, find the angle C. (P. U. 1941)

92. Case II. *To solve a triangle, given two sides and the included angle.*

Let b, c, A be given. Let b be greater than c. From the formula

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{b-c}{b+c}\tan\left(90^\circ-\frac{A}{2}\right)$$

we get, by taking logarithms

$$\log\tan\frac{B-C}{2}=\log(b-c)-\log(b+c)+\log\tan\left(90^\circ-\frac{A}{2}\right).$$

from which we find $\dfrac{B-C}{2}$.

Also $\dfrac{B+C}{2}=90^\circ-\dfrac{A}{2}$, which gives $\left(\dfrac{B+C}{2}\right)$.

By adding and subtracting, we get B and C.

The side a can be found from the formula

$$\frac{a}{\sin A}=\frac{b}{\sin B}, \text{ from which } a=\frac{b \sin A}{\sin B};$$

hence $\log a=\log b+\log \sin A-\log \sin B$, so that a is known.

Ex. 1. Given $b=130$, $c=72$ and $A=42°$, solve the triangle.

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{130-72}{130+72}\cot 21°=\frac{29}{101}\tan 69°$$

$$\therefore \log\tan\frac{B-C}{2}=\log 29-\log 101+\log\tan 69°$$

$$=1{\cdot}4624-2{\cdot}0043+{\cdot}4158$$
$$=-{\cdot}1261=\bar{1}{\cdot}8739.$$

$$\therefore \quad \frac{B-C}{2}=36°\ 47'\ 46''.$$

Also $\dfrac{B+C}{2}=90°-\dfrac{A}{2}=69°.$

Adding we get $B=105°\ 47'\ 46''$.

Subtracting, $C=32°\ 12'\ 46''$,

Again $a=\dfrac{b \sin A}{\sin B}=\dfrac{130 \sin 42°}{\sin 105°\ 47'\ 46''}$.

$$\therefore \log a=\log 130+\log\sin 42° -\log\sin 105°\ 47'\ 46''$$
$$=2{\cdot}1139+\bar{1}{\cdot}8255-\log\sin 74°\ 12'14''$$
$$=2{\cdot}1139+\bar{1}{\cdot}8255-\bar{1}{\cdot}9833$$
$$=1{\cdot}9561$$
$$\therefore \quad a=90{\cdot}38.$$

But if the Principle of Proportional Parts is not used, then

$$B=105°\ 48',\ C=32°\ 12',$$

and $\quad a=90{\cdot}38.$

Ex. 2. Given $b=68$, $c=27$, $B-C=70°$, solve the triangle.

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$\therefore \quad \tan 35^\circ = \dfrac{68-27}{68+27}, \quad \dfrac{1}{\tan\dfrac{A}{2}} = \dfrac{41}{95}. \quad \dfrac{1}{\tan\dfrac{A}{2}}$

i.e., $\quad \tan\dfrac{A}{2} = \dfrac{41}{95}. \quad \dfrac{1}{\tan 35^\circ}.$

$\therefore \quad \log\tan\dfrac{A}{2} = \log 41 - \log 95 - \log\tan 35^\circ$

$= 1{\cdot}6128 - 1{\cdot}9777 - \bar{1}{\cdot}8452$

$= 2 - {\cdot}2107 = \bar{1}{\cdot}7899.$

$\therefore \quad \dfrac{A}{2} = 31^\circ\ 39'$, *i.e.*, $A = 63^\circ\ 18'$.

$\therefore \quad B+C = 180^\circ - 63^\circ\ 18' = 116^\circ\ 42'.$

Also $\quad B-C = 70^\circ$;

$\therefore \quad 2B = 186^\circ\ 42' \qquad \therefore\ B = 93^\circ\ 21'$

and $\quad 2C = 46^\circ\ 42' \qquad \therefore\ C = 23^\circ\ 21'$

Now, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}. \qquad \therefore\ a = \dfrac{b\sin A}{\sin B}$

$\therefore \quad \log a = \log b + \log\sin A - \log\sin B$

$= \log 68 + \log\sin 63^\circ\ 18' - \log\sin 93^\circ\ 21'.$

$= 1{\cdot}8325 + \bar{1}{\cdot}9510 - \log\sin 86^\circ\ 39'$

$= 2{\cdot}7835 - 1{\cdot}9992$

$= 1{\cdot}7843$

$\therefore \quad a = 60{\cdot}85.$

EXERCISE XXXII

Solve the following triangles and check the solutions.

1. $b=11$, $c=9$, $A=32^\circ\ 30'$.
2. $a=29{\cdot}8$; $c=32{\cdot}42$; $B=26^\circ\ 14'$.
3. $b=52{\cdot}92$; $c=36{\cdot}04$; $A=62^\circ\ 17'$.
4. $a=872{\cdot}5$; $b=632{\cdot}7$; $C=80^\circ$. (P. U.).
5. $b=82{\cdot}9$; $c=251$; $A=60^\circ$; find B and C.
6. $b=27$; $c=33{\cdot}48$; $A=60^\circ$: find B and C.
7. $a=17{\cdot}6$; $b=24{\cdot}03$. $C=121^\circ\ 38'$.
8. $a=681$; $B=50^\circ\ 42'$ $b=243$.
9. Solve the triangle in which $A=42^\circ\ 54'$, $b=25{\cdot}07$, $c=26^\circ\ 55'$.
10. Solve the triangle in which $b=37{\cdot}2$, $c=22{\cdot}3$; $A=29^\circ\ 38'$.
11. Area$=2457$; $a=79$, $c=97$,

12. Two sides of a triangle are in the ratio 16 : 9 and the included angle is 102° 48′. Find the other angles.

13. AOB is an equilateral triangle and C and D are points in AB such that AC = CD = DB = $2a$. Find by calculation the amount by which the angle COD exceeds one third of the angle AOB.

93. Case III. *Given one side and two angles, viz., a, B, C ; to solve the triangle.*

The third angle A can be found by subtracting the sum of B and C from 180°.

The sides b and c are obtained from the relations

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}, \text{ giving}$$

$$b=\frac{a \sin B}{\sin A} \text{ and } c=\frac{a \sin C}{\sin A}, \text{ whence}$$

$$\log b=\log a+\log \sin B-\log \sin A,$$

$$\text{and } \log c=\log a+\log \sin C-\log \sin A,$$

so that b and c are obtained with the help of the tables.

Ex. 1. Solve the triangle when B = 64° 23′ ; C = 72° 43′ and a = 18·92.

$$A=180°-(64° 23'+72° 43')=42° 54'.$$

$$b=\frac{a \sin B}{\sin A}=\frac{18\cdot92 \sin 64° 23'}{\sin 42° 54'}$$

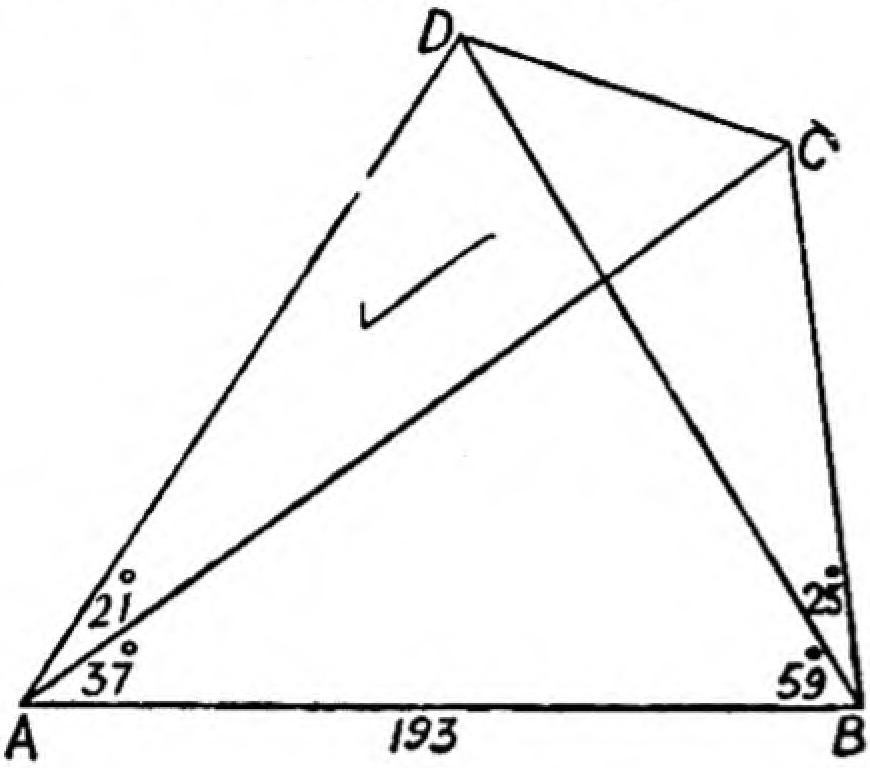
$$\therefore \log b=\log 18\cdot92+\log \sin 64° 23'-\log \sin 42° 54'$$
$$=1\cdot2770+\bar{1}\cdot9551-1\cdot8330$$
$$=1\cdot3991.$$
$$\therefore \quad b=95\cdot07.$$

$$\text{Again, } c=\frac{18\cdot92 \sin 72° 43'}{\sin 42° 54'}$$

$$\therefore \log c=\log 18\cdot92+\log \sin 72° 43'-\log \sin 42° 54'$$
$$=1\cdot2770+\bar{1}\cdot9799+\bar{1}\cdot8330$$
$$=1\cdot4239$$
$$\therefore \quad c=26\cdot55.$$

Ex. 2. In the quadrilateral ABCD, AB = 193, ∠BAC = 37°, ∠CAD = 21°, ∠ABD = 59°, ∠CBD = 23°, find CD.

From triangle ABD,



$$\frac{AD}{\sin 59^\circ} = \frac{123}{\sin 63^\circ}$$

$\therefore$ log AD = log 193 + log s.n 59° − log sin 63°

$= 2{\cdot}2866 + \bar{1}{\cdot}9331 - \bar{1}{\cdot}9499$

$= 2{\cdot}2688$

$\therefore$ AD = 185·7.

From triangle ABC

$$\frac{AC}{\sin 82^\circ} = \frac{193}{\sin 61^\circ}$$

$\therefore$ log AC = log 193 + log sin 82° - log sin 61°

$= 2{\cdot}2856 + \bar{1}{\cdot}9958 - \bar{1}{\cdot}9418$

$= 2{\cdot}339$ $\quad\therefore$ AC = 218·6.

Now from triangle ACD

$$\tan\frac{D-C}{2} = \frac{d-c}{d+c}\cot\frac{A}{2} = \frac{32{\cdot}9}{404{\cdot}3}\tan 79^\circ\, 30' \quad \therefore \log\tan\frac{D-C}{2}$$

$= \log 32{\cdot}9 - \log 404{\cdot}3 + \log\tan 79^\circ\, 30'$

$= 1{\cdot}5172 - 2{\cdot}6067 + {\cdot}7320$

$= \bar{1}{\cdot}6425.$

$\therefore \quad \frac{D-C}{2} = 23^\circ\, 42'$ approximately and $\frac{D+C}{2} = 79^\circ\, 39'$.

$\therefore \quad D = 103^\circ\, 12'$ and $C = 55^\circ\, 48'$.

Also $\quad \dfrac{CD}{\sin 21^\circ} = \dfrac{185{\cdot}7}{\sin 55^\circ\, 48'}$,

$\therefore$ log CD = log 185·7 + log sin 21° − log sin 55° 48′.

$= 1{\cdot}9056.$ $\quad\therefore$ CD = 80·46.

EXERCISE XXXIII

Solve the following triangles and check the solutions:

1. $a = 226{\cdot}9$; $B = 73^\circ\, 35'$; $C = 39^\circ\, 45'$.

2. Base = 7 and base angles are 129° 23′ and 38° 36′. Find the length of the shorter side.

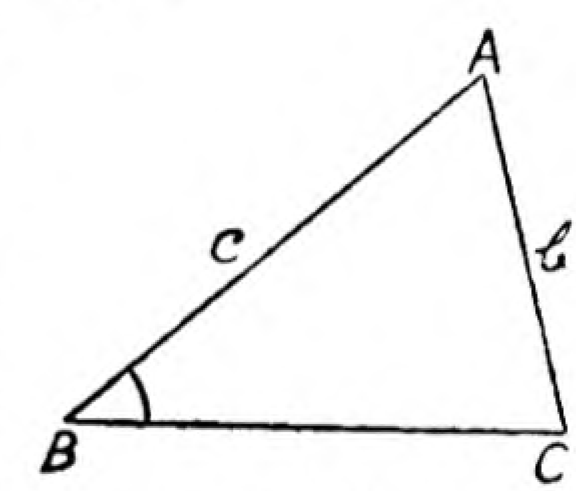
3. Two angles are 180° 20′ and 11° 40′ and the longest side is 1000 ft. Find the shortest side.

4. $A=72^\circ\ 19'$; $B=83^\circ\ 17'$; $c=92{\cdot}93$.
5. $B=64^\circ\ 23'$; $C=72^\circ\ 43'$ and $a=18{\cdot}92$.
6. $B=118^\circ\ 37'$; $C=31^\circ\ 45'$, $a=20{\cdot}95$.
7. $A=66^\circ\ 38'$; $B=26^\circ 14'$ and $c=32{\cdot}42$.

8. A and B are two points 100 ft. apart on the same bank of a straight river; C is a point on the opposite bank and the angles CAB and CBA are found to be 47° and 56° respectively. Calculate to the nearest foot the width of the river.

9. Two men are stationed at A and B respectively, and they observe a point C; the first man finds that the angle BAC is 47° 22′, and the second man finds that the angle ABC is 63° 5′. If AB=100 feet, how far is C from A ?

94. Case IV. *Given two sides, b, c, and the angle B opposite to one of those sides, to solve the triangle.*



Angle C may be found from the relation $\dfrac{\sin C}{c}=\dfrac{\sin B}{b}$ so that $\log \sin C = \log c + \log \sin B - \log b$, which gives C. Let one value be x°, then $180^\circ - x^\circ$ is another value of C, which satisfies the equation. Any of these two values is inadmissible if, when added to the given angle B, the sum is greater than 180°; and $A=180^\circ - B - C$, so that A has two values if the two values of C found above are admissible.

The third side a may be found from the relation

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}$ so that $\log a = \log b + \log \sin A - \log \sin B$ which gives a, there being two values for a, in case A has two values.

Ex. 1. Solve the following: $b=82{\cdot}5$; $c=182{\cdot}5$; and $C=72^\circ\ 15'$.

$$\frac{\sin B}{b}=\frac{\sin C}{c} \qquad \therefore \quad \sin B=\frac{b \sin C}{c}$$

$$\begin{aligned}\therefore \quad \log \sin B &= \log b + \log \sin C - \log c\\ &= \log 82{\cdot}5 + \log \sin 72^\circ\ 15' - \log 182{\cdot}5\\ &= 1{\cdot}9165 + \bar{1}{\cdot}9788 - 2{\cdot}2613\\ &= \bar{1}{\cdot}6340.\end{aligned}$$

$\therefore \quad B=25^\circ 30'$ or $154^\circ 30'$. $\quad \therefore \quad \sin(180^\circ - B)=\sin B.$

The obtuse value of B is inadmissible because when it is added to C, the sum is greater than 180°.

$A=180^\circ-B-C=180^\circ-(25^\circ 30'+72^\circ 15')=82^\circ 15'.$

Now $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$.

$$\begin{aligned}\therefore \quad \log a &= \log c+\log \sin A-\log \sin C\\ &= \log 182.5+\log \sin 82^\circ 15'-\log \sin 72^\circ 15'\\ &= 2.2613+\bar{1}.9961-\bar{1}.9788\\ &= 2.2785 \quad \therefore \quad a=190.\end{aligned}$$

Ex. 2. Solve the following, $a=82.31$, $c=72.95$ and $C=42^\circ 27'$.

$$\frac{\sin A}{a}=\frac{\sin C}{c} \qquad \therefore \quad \sin A=\frac{a \sin C}{c}.$$

$$\begin{aligned}\therefore \quad \log \sin A &= \log a+\log \sin C-\log c\\ &= \log 82.31+\log \sin 42^\circ 27'=\log 72.95\\ &= 1.9155+\bar{1}.8293-1.8630\\ &= \bar{1}.8818.\end{aligned}$$

$\therefore \quad A=49^\circ 37'$ or $130^\circ 23'$.

Both the values of A are admissible because the sum of the obtuse value of A and the value of C is not greater than 180°.

Let the acute value of A be called A_1 and the obtuse value A_2

$B_1=180^\circ-A_1-C=180^\circ-49^\circ 37'-42^\circ 27'=87^\circ 56'$ and $B_2=180^\circ-A_2-C=180^\circ-130^\circ 23'-42^\circ 27'=7^\circ 10'$.

Now there will be two values of b, say b_1 and b_2 ;

$$\frac{b_1}{\sin B_1}=\frac{c}{\sin C}$$

$$\begin{aligned}\log b &= \log c+\log \sin B_1-\log \sin C\\ &= \log 72.25+\log \sin 87^\circ 56'-\log \sin 42^\circ 27'\\ &= 1.8630+\bar{1}.9997-\bar{1}.8293\\ &= 2.0334.\end{aligned}$$

$\therefore \quad b_1=108.$

$$\begin{aligned}\text{Similarly } \log b_2 &= \log c+\log \sin B_2-\log \sin C\\ &= \log 72.95+\log \sin 7^\circ 10'-\log \sin 42^\circ 27'\\ &= 1.8630+\bar{1}.9958+\bar{1}.8293.\end{aligned}$$

$=1{\cdot}1295$

$\therefore \quad b_2=13{\cdot}48.$

EXERCISE XXXIV

Solve the following triangles and check the solutions :

1. In a $\triangle$ if $a=3$, $b=3\sqrt{3}$, $A=30°$, find B. (C. U.)
2. $b=9463$, $c=7590$, $C=43°\ 47'$.
3. $a=13{\cdot}3$, $b=8{\cdot}7$, $B=33°\ 20'$, find A.
4. $B=30°$, $c=924{\cdot}3$, $b=123{\cdot}4$.
5. $A=20°\ 41'$ $b=137$, $a=115$.
6. $a=324{\cdot}7$, $c=421{\cdot}7$, $C=35°$.
7. $a=942$, $b=1413$, $A=40°$.
8. $a=52{\cdot}48$, $b=27{\cdot}24$, $A=56°\ 28'$.
9. $a=30{\cdot}28$, $b=21{\cdot}85$, $B=46°\ 12'$.
10. $a=342{\cdot}9$, $b=745{\cdot}9$, $A=43°\ 35'$.
11. In a triangle, $A=94°\ 16'$, $b=5038$, $c=6840$.

    Find B and C. (P. U. 1938)

    [Notice here since $c>b$ Napier's analogy in the form $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$ is not applicable. Use it in the form $\tan\dfrac{C-B}{2}=\dfrac{c-b}{c+b}\cot\dfrac{A}{2}$.]

12. $a=42{\cdot}24$, $b=47{\cdot}75$; $A=21°\ 6'$. (P. U. 1942)

95. Discussion of the Ambiguous Case.

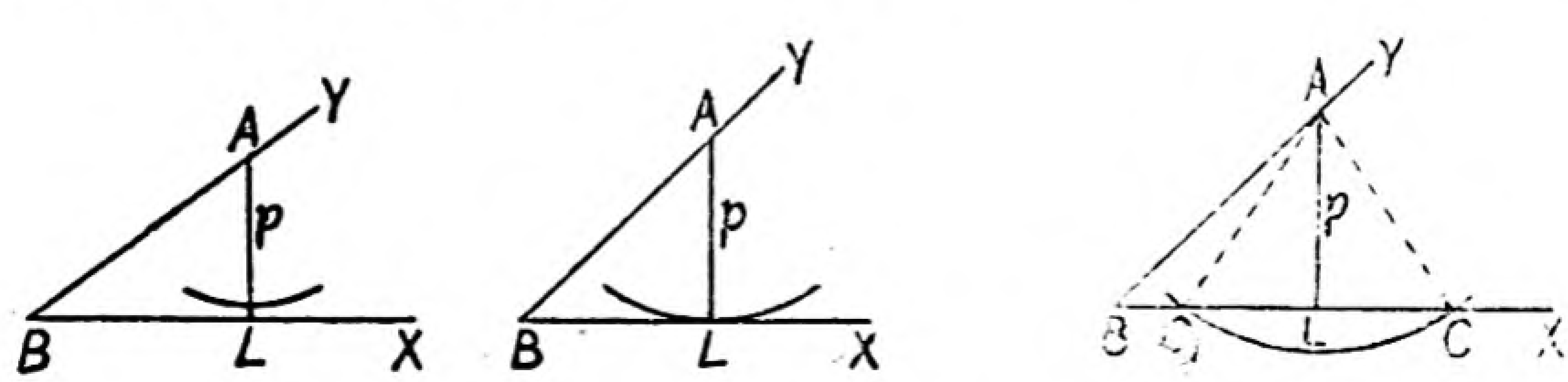
In case IV above, we have seen that with the given data sometimes one triangle is possible, sometimes two and sometimes none. We shall discuss this case now in detail.

Geometrical Discussion.

Let b, C, B be given.

(a) Firstly, let B be acute.

Take the angle $XBY=\angle B$, along BY cut off $BA=c$. From A draw $AL=(p)\perp BX$. With centre A and radius equal to b, draw a circle. Then



(*i*) if $b<p$, the circle does not cut BX and hence no triangle is possible.

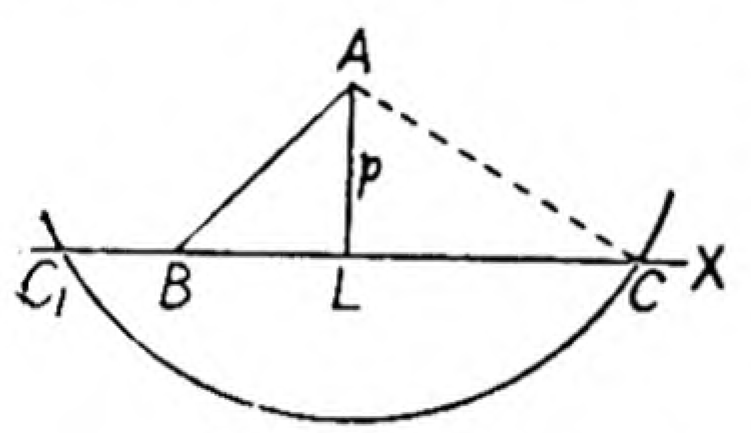
(*ii*) if $b=p$, the circle touches BX at L and only one triangle (right-angled) is possible.

(*iii*) if $b>p$ there are three sub-cases to be considered:

1. When $b<c$, the circle cuts BX into two points C and C_1. In this case two triangles ABC and ABC_1 are possible with the given data.

This is called the **Ambiguous Case**.

2. When $b=c$ the circle will cut BX in two points, but one of these points coincides with B and hence in this case also only one triangle is possible.



3. When $b>c$, this circle cuts BX in two points which are however on opposite sides of B. Hence in this case also only one triangle can be drawn with the given data.

(*b*) Secondly, let B be obtuse.

Proceeding as above, the reader will find that there is no triangle possible except when $b>c$.

Trigonometrical Discussion.

$$\sin C = \frac{c \sin B}{b} \qquad \ldots\ldots(i)$$

(*a*) Firstly, let B be acute. Then

(*i*) If $b<c \sin B$, $\sin C$ is greater than unity, which is impossible, hence no triangle is possible in this case.

(*ii*) If $b=c \sin B$, $\sin C=1$; therefore $C=90°$. Hence only one triangle (right-angled) is possible.

(*iii*) If $b>c \sin B$, $\sin C<1$; therefore there are two values of C having $\frac{c \sin B}{b}$ as their sine, one acute (say C_1) and the other obtuse (say C_2). Both of them are not however, always admissible. Then sub-cases arise:

1. When $b<c$, then $B<C$, and therefore C may either

be acute or obtuse, so that both values of C are admissible. This is known as the Ambiguous Case.

2. When $b=c$, then $B=C$. Only the acute value of C is valid, hence there is only one triangle possible.

3. When $b>c$, then $B>C$. Therefore B being acute, C must also be acute. Hence only one triangle is possible.

(*b*) Secondly, let B be obtuse.

If b be $<$ or $=c$, then B is $<$ or $=C$, so that in both cases C would be obtuse. As a triangle cannot have two obtuse angles, the triangle would be impossible.

If $b>c$, then $B>C$, Hence the acute value of C would be possible and not the obtuse. Hence there is only one solution.

Algebraical Discussion.

We have $b^2=c^2+a^2-2ac\cos B$.

$\therefore\quad a^2-2ac\cos B+(c^2-b^2)=0$.

By solving the quadratic in a, we obtain

$$a=c\cos B\pm\sqrt{b^2-c^2\sin^2 B}$$

(*a*) Firstly, let B, be acute. Then

(*i*) If $b<c\sin B$, the quantity under the radical is negative so that $\sqrt{b^2-c^2\sin^2 B}$, is imaginary. Thus in this case a is imaginary, and hence no triangle is possible.

(*ii*) If $b=c\sin B$, the quantity under the radical is zero. Thus in this case there is only one value of a, *i.e.*, $a=c\cos B$ which gives $\cos B=\frac{a}{c}$ showing that C is a right angle.

Thus only one triangle is possible and it is right-angled.

(*iii*) If $b>c\sin B$, there are two values of a. But since a must be positive, the lower sign in the value of a would give the positive result only when

$\sqrt{b^2-c^2\sin^2 B}<c\cos B$

i.e., $b^2-c^2\sin^2 B<c^2\cos^2 B$

i.e., $b^2<c^2(\sin^2 B+\cos^2 B)$

i.e., $b^2<c^2$, *i.e.*, $b<c$.

Thus there are two triangles possible only when $b>c\sin B$ and $b<c$.

(*b*) Secondly, let B be obtuse. Then $c\cos B$ is negative.

Thus one value of a obtained above is always negative. Positive value for a would only be obtained when

$b \cos B + \sqrt{b^2 - c^2 \sin^2 B}$ is positive

i.e., $b^2 - c^2 \sin^2 B > c^2 \cos^2 B$

i.e., $b^2 > c^2 \sin^2 B + c^2 \cos^2 B$

i.e., $b^2 > c^2$ or $b > c$.

Thus when B is obtuse, a triangle is possible only when $b > c$.

From each of the foregoing discussions it follows that the only case in which an ambiguous solution can arise, if it arises at all, is when the smaller of the two given sides is opposite to the given angle.

Thus, given, b, c, B, two triangles are possible only when (*i*) $b > c \sin B$ and (*ii*) $b < c$.

96. *Other cases in which a triangle may be solved.*

The cases which have already been examined may be called the four standard cases in the solution of triangles. However a triangle may be determined in various other ways; as for example a triangle is fixed when its base, its height and one of its angles be given; or in general, a triangle is fixed when any three independent quantities connected with it are fixed, provided that at least one of the quantities is a length.

We shall illustrate this fact by a few examples.

Ex. 1. Given the base, one of the base angles, and the height of the triangle, show how to solve the triangle.

Let h be the height, a the base and B the base angle given.

Then $h = c \sin B$ so that c is known. Thus two sides a and c and the included angle B are known and the triangle can be solved.

Ex. 2. Given the perimeter and two angles, show how to solve the triangle.

Let the given perimeter be $2s$ and let the given angles be B and C.

Now $A = 180° - (B + C)$, so that A is known.

Also since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \dfrac{a+b+c}{\sin A + \sin B + \sin C}$

$$=\frac{2s}{\sin A+\sin B+\sin C}.$$

Therefore $a=\dfrac{2s\sin A}{\sin A+\sin B+\sin C}$ and similar expressions for b and c.

Ex. 3. Given the three altitudes of a triangle, show how to solve the triangle.

Let, p, q, r be the perpendiculars from A, B, C on the opposite sides and S the area of triangle.

Then $S=\frac{1}{2}ap=\frac{1}{2}bq=\frac{1}{2}cr$, so that

$$a=\frac{2S}{p},\quad b=\frac{2S}{q},\quad c=\frac{2S}{r},$$

and therefore $s=S\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$.

Substituting these values of s, a, b, c in

$\tan\frac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$ and two other similar equations, we get A, B and C, the unknown S cancelling out.

Thus the three angles A, B, C are known.

Also $p=c\sin B$, so that c is known. Similarly a and b are known.

Ex. 4. Given the base, the difference between the two sides and the vertical angle of a triangle, show how to solve it.

Let a, $b-c$ and A be given.

Then we know that $\dfrac{b-c}{a}=\dfrac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}$,

$\therefore\ \log\sin\frac{B-C}{2}=\log(b-c)-\log a+\log\sin\left(90^\circ-\frac{A}{2}\right)$

Thus $\frac{B-C}{2}$ is known.

Also $\frac{B+C}{2}$, being equal to $90^\circ-\frac{A}{2}$ is known.

Hence B and C can be found.

The sides b and c can be found by using the relations $\frac{b}{\sin B}=\frac{a}{\sin A}$ and $\frac{c}{\sin C}=\frac{a}{\sin A}$. Thus the triangle is completely solved.

EXERCISE XXXV

1. In the triangle ABC
$A=110°$, $a=500$, $b-c=60$, find B and C.

2. Show how to solve a triangle when two angles and one of the medians are given.
If $A=58° 44'$, $C=73° 38'$ and the median $AD=400$ inches, find the side AB of the triangle.

3. Show how to solve a triangle having given the base, the height and the difference of the base angles, the base angles being both acute.

4. Given a, $b+c$ and A, show how to solve the triangle ABC.

REVISION QUESTIONS IX

1. Show how to solve a triangle when its three sides are given.
In a triangle, $a=18$, $b=20$, $c=22$.
Calculate the value of $L \tan \frac{A}{2}$.

2. The sides of a triangle are 2, 3, 4. Find its greatest angle.

3. Solve the triangle, given
$$a=52{\cdot}48,\ b=27{\cdot}24,\ c=62{\cdot}37.$$

4. Given B, c, a, show how to solve the triangle.
Two sides of a triangle are 3 and 5, and the included angle is 120°. Find the other angles.

5. Given $b=540$, $c=420$, $A=52° 6'$; find B and C.

6. The sides of a triangle are 50, 36, 28. Find the greatest angle.

7. In any triangle show that
$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$
Two sides of a triangle are 1 and 3, and the included angle is 40°. Find the other angles.

8. Can a triangle be found in which $a=118$, $b=235$ and $A=31° 8'$?

9. If a, b, A be given and $a<b$, and if c, c' be the two values of the third side, then

$$c^2-2cc' \operatorname{Cos} 2A+c'^2=4a^2 \cos^2 A.$$

10. In a triangle, A, B, c are given. Show how to solve it.

$A=33° 40'$, $B=101° 48'$, $c=40{\cdot}27$. Solve the triangle.

11. In a triangle ABC the parts, a, b, A are given, and $a<b$, $a>b \sin A$; prove that there are two solutions ; and that if c, c' be the two values of the third side, then $c+c'=2b\times\cos A$, and $cc'=b^2-a^2$.

CHAPTER XIII

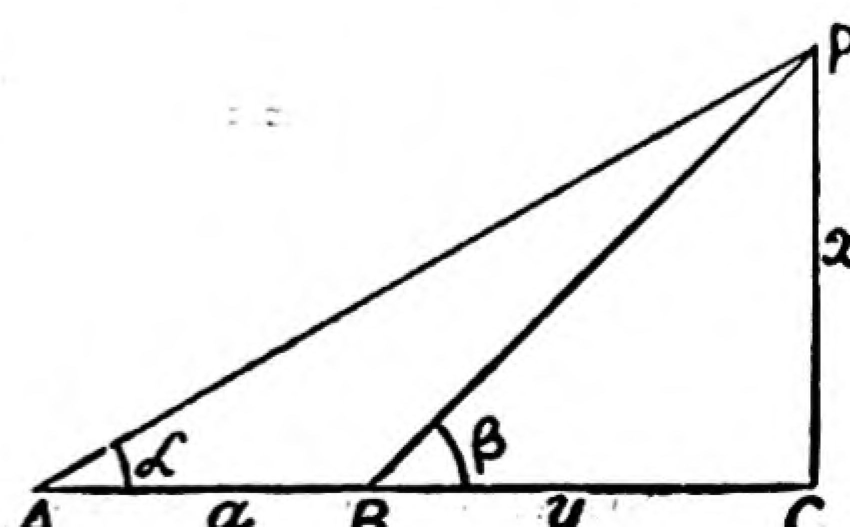
HEIGHTS AND DISTANCES

97. We have already seen some of the simple applications of elementary trigonometry to the measurement of heights and distances.

We are now in a position to take up a few more applications of trigonometry and show in what way trigonometry helps the surveyor, the civil engineer, the military engineer or the map-maker.

This will be best illustrated by the solved examples that follow.

Ex. 1. A person, wishing to find the height of an inaccessible vertical pillar standing on the horizontal plane, observes that at a certain point A the angle of elevation of the top of the pillar is α; after walking a distance a towards the foot of the pillar he finds that the angle of elevation is β. Calculate the height of the tower and its distance from the second position of the man.



Let CP be the pillar and let A and B be the two positions of the man so that $AB=a$.

Let $CP=x$ and $BC=y$.

From the triangle APB,

$$\frac{PB}{\sin \alpha}=\frac{AB}{\sin (\beta-\alpha)}.$$

$$\therefore \quad PB = \frac{a \sin \alpha}{\sin (\beta - \alpha)}.$$

Hence $x = PB \sin \beta = \dfrac{a \sin \alpha}{\sin (\beta - \alpha)} \sin \beta.$

and $\quad y = PB \cos \beta = \dfrac{a \sin \alpha}{(\sin \beta - \alpha)} \cos \beta.$

Thus x and y are determined by means of expressions suitable for logarithmic calculations.

Note.—This gives us a method of finding the height and the distance of an inaccessible object on a horizontal plane.

Another method for finding the height is explained in Example 3.

Ex. 2. In example 1, if $\beta = 55°$, $\alpha = 25°$, and $a = 100$ feet, find the height x.

$$x = \frac{a \sin \alpha}{\sin (\beta - \alpha)} \sin \beta = \frac{100 \sin 25°}{\sin 30°} \sin 55°.$$

$$\therefore \log x = \log 100 + \log \sin 25° + \log \sin 55° - \log \sin 30°$$
$$= 2 + \bar{1}\cdot6259 + \bar{1}\cdot9134 - \bar{1}\cdot6990 = 1\cdot8403$$

$\therefore \quad x = 69\cdot23$ feet, nearly.

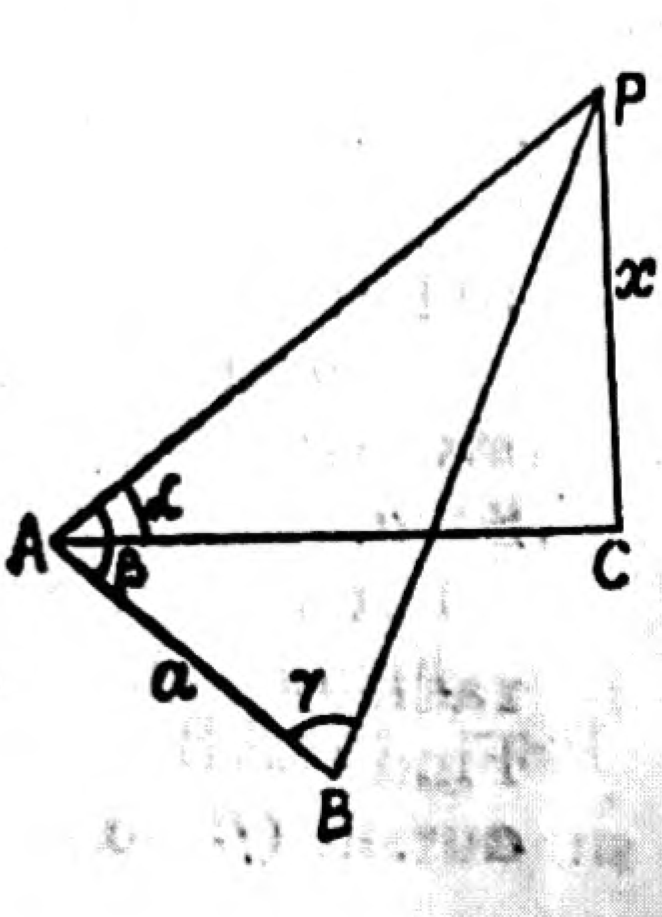
Ex. 3. A person wishing to find the height of a vertical pillar standing on a horizontal plane observes that at a point A in the horizontal plane the angle of elevation of the top P of the pillar is α. Then he walks to a point B, AB being a feet, and $\angle PAB$ being β, and finds that $\angle PBA$ is γ. Find the height of the pillar.

From triangle APB we have

$$\frac{AP}{a} = \frac{\sin ABP}{\sin APB}$$

$$= \frac{\sin \gamma}{\sin (180° - \beta - \gamma)}$$

$$= \frac{\sin \gamma}{\sin (\beta + \gamma)}$$



so that $AP = \dfrac{a \sin \gamma}{\sin (\beta + \gamma)}.$

Now from triangle PAC, we have

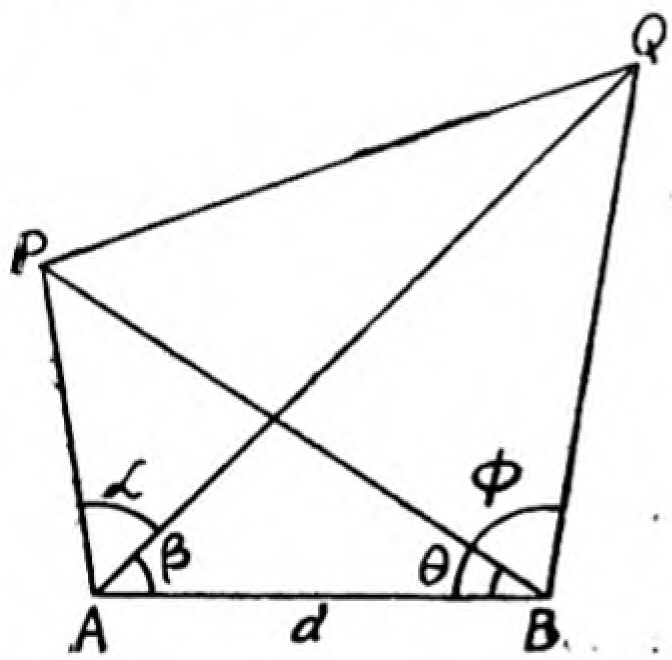
$$x=\text{AP}\ \sin\alpha=\frac{a\sin\gamma}{\sin(\beta+\gamma)}\sin\alpha.$$

Hence x is determined by a formula suitable for logarithmic calculations.

Note that the above figure is not in one plane. In fact PAC and PAB are triangles in two different planes.

Note.—This gives another method for finding the height of an inaccessible object standing on a horizontal plane.

Ex. 4. A person wishing to find the distance between two inaccessible objects P and Q measures a distance AB $=a$ feet. At A he finds that $\angle$PAQ $=\alpha$, $\angle$QAB $=\beta$ and $\angle$PAB $=\gamma$; at B he finds that $\angle$PBA $=\theta$ and $\angle$QBA $=\phi$. Indicate a method for finding the distance PQ suitable for logarithmic calculations.



From triangle PAB, we have

$$\frac{\text{AP}}{a}=\frac{\sin \text{PBA}}{\sin \text{BPA}}=\frac{\sin\theta}{\sin(180^\circ-\theta-\gamma)}$$

$$=\frac{\sin\theta}{\sin(\theta+\gamma)},$$

so that $\text{AP}=\dfrac{a\sin\theta}{\sin(\theta+\gamma)}.$

Similarly from triangle QAB we have

$$\frac{\text{AQ}}{a}=\frac{\sin \text{ABQ}}{\sin \text{BQA}}=\frac{\sin\phi}{\sin(180^\circ-\beta-\phi)}$$

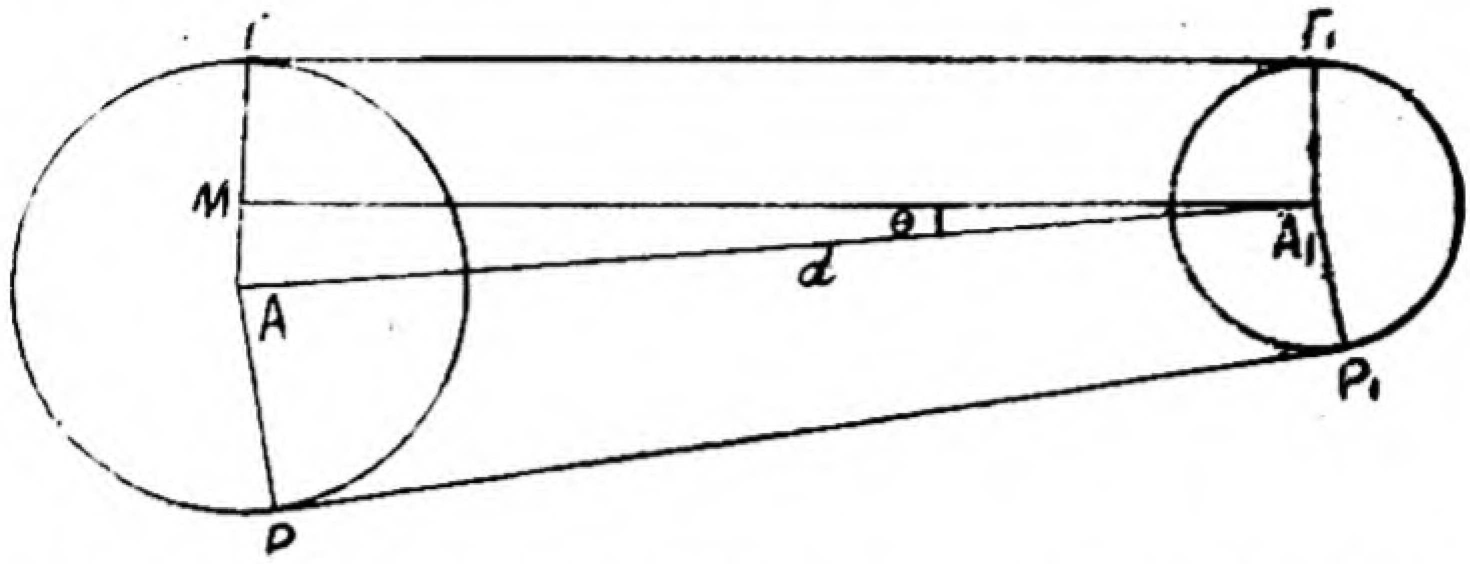
$$=\frac{\sin\phi}{\sin(\beta+\phi)}\ \text{so that AQ}=\frac{a\sin\phi}{\sin(\beta+\phi)}.$$

Thus AP and AQ are determined by expressions suitable for logarithmic calculations.

Now in triangle PAQ, AP and AQ are known and the included angle PAQ is known, the triangle can therefore be solved by the method of Art. 91, so that PQ is determined.

Ex. 5. An endless belt passes over two pulleys, whose radii are R and r and whose centres are distant d apart. Find the length of the belt, assuming it to be taut throughout.

The belt may be open or crossed. In the former case the two pulleys rotate in the same direction and in the latter case they rotate in opposite directions.



The figure is drawn for the case when the belt is open.

AT, A_1T_1, AP and A_1P_1 are radii to the last points of contact of the belt and the pulleys. AT and A_1T_1 are parallel and so are AP and A_1P_1; also A_1M is drawn perpendicular to AT. $AM=R-r$.

Let the angle AA_1M be θ radians so that

$\sin\theta=\dfrac{AM}{AA_1}=\dfrac{R-r}{d}$. This determines θ.

Now $\angle A_1AT=\dfrac{\pi}{2}-\theta$ and therefore angle at the centre subtended by the portion of the belt in contact with the larger pulley is $2\pi-2\left(\dfrac{\pi}{2}-\theta\right)=\pi+2\theta$ and $\therefore$ length of belt in contact is $R(\pi+2\theta)$. Similarly length of belt in contact with the smaller pulley is $r(\pi-2\theta)$. Also $TT_1=MA_1=d\cos\theta$.

Hence total length of belt is $R(\pi+2\theta)+r(\pi-2\theta)+2d\cos\theta$.

$$=\pi(R+r)+2\theta(R-r)+2(R-r)\cot\theta.$$

If the belt is crossed it can be similarly shown that its length is

$$\pi(R+r)+2(R+r)(\theta+\cot\theta).$$

EXERCISE XXXVI

1. A statue on the top of a pillar subtends the same angle α, at distances 9 and 11 yds. from the pillar. If $\tan\alpha=\frac{1}{10}$, find the height of the statue. [P. U. 1920]

2. The angles of elevation of a building as seen from points B and C are respectively 55° and 25°, the points B and C being at a distance of 100 feet from one another in a horizontal straight line which if produced could pass through the base of the building. Find the height of the building.

[*P. U. 1920*]

3. From the two points A and B one mile apart on a horizontal plane the angles of elevation of the top C of a mountain are found to be 26° and 41° respectively, A, B, C, being in a vertical plane. Find the height of the mountain.

4. A man observes the elevation of a tower to be 15°, he walks directly towards it for a distance $2a$ when he finds the elevation to be 75°. Show that the height of the tower is $\frac{a}{\sqrt{3}}$.

5. A tower subtends an angle α at a point on the same level as the foot of the tower, and at a point h feet above the first the angle of depression of the foot of tower is β. Find the height of the tower.

6. Three stations A, B, C, are in a horizontal line passing through the foot of a tower, and the angles of elevation of the top of the tower at the three points are found to be θ, $90°-\theta$, 2θ respectively.

If $AB=a$, $BC=b$, prove that

$a=2(a+b)\cos 2\theta$ and that the height of tower is

$\frac{1}{2}\sqrt{(3a+2b)(a+2b)}$, $(30°<\theta<45°)$. [*B.U. 1929*]

7. An observer sees due North at an elevation of 30° and at a height p above the ground an aeroplane travelling horizontally due East. If the speed of the aeroplane is v miles per hour, prove that its angle of elevation as seen by he observer after k hours is

$$\sin^{-1}\frac{p}{\sqrt{4p^2+k^2v^2}}.$$

8. The angle of elevation of the top of a vertical tower standing on a horizontal plane is observed at a point A in the plane to be $\tan^{-1}\frac{10}{3}$. AB is drawn in the plane at right angles to the line which joins A to the base of the tower and 40 ft. in length and the elevation of the top of the tower from B is $\tan^{-1} 2$. Find the height of the tower.

9. The angle of elevation of a tower from a point A due south of it is x and from a point B due east of A is y. If $AB=l$, show that the height h of the tower is given by $h^2(\cot^2 y-\cot^2 x)=l^2$.

10. A ring, diameter 2 feet, is suspended by six equal strings from a point 6 inches above its centre, the strings being attached at equal intervals round its circumference. Find the length of each string and the angle between consecutive strings.

11. From the top of a cliff the known distance a between two buoys is observed to subtend an angle θ, while their depressions are α and β; prove that the height of the cliff above the sea is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{\sin^2\alpha+\sin^2\beta-2 \sin \alpha \sin \beta \cos \theta}}.$$

REVISION QUESTIONS X

1. A path up a hillside consists of two straight portions AB and BC; AB is 70 yards long and is inclined at 19° to the horizontal; BC is 30 yards long and is inclined at 16° to the horizontal. Calculate to the nearest yard the vertical height of C above A.

2. The angles of elevation of the top of a tower from the top and the bottom of a building h feet high are α and β respectively. Find an expression for the height of the tower suitable for logarithmic calculation. Find the height of the tower when $h=250$ feet, $\alpha=50°$ and $\beta=75°$.

3. ABCD is a rectangular courtyard on level ground, and AB=400 ft., BC=300 ft., AP is a vertical tower on which stands a vertical flagstaff PQ. If angle PCA=16° 42′, and angle QCA=19° 17′, calculate AP, AQ and hence find the height of the flagstaff.

4. ABCD is the floor of a room of rectangular plane and of height 10 ft.; X, Y are the points of the ceiling that are vertically above A and B respectively. If AD=15 ft. and tan XBA=$\frac{1}{2}$, find the angle BDY.

5. A person in a balloon observes that the angles of depression of the top and the bottom of a tower h feet high are α and β. Find expressions suitable for logarithmic cal-

culation for the height of the balloon and its horizontal distance from the foot of the tower.

If $h=200$, $\alpha=60°$, and $\beta=70°$, find the height of the balloon.

6. A and B are points at a distance a apart on a straight road running East and West, and B being East of A. A point P is α North of East from A and β North West from B. Show that the perpendicular distance of P from the road is

$$\frac{a \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}.$$

Calculate this distance when $a=950$ yards, $\alpha=37° 30'$, $\beta=70° 40'$. [Hint: This can be put in the form

$\dfrac{a \sin \alpha \sin \beta}{\sin (\alpha+\beta)}$ which is suitable for logarithmic calculation.]

7. A vertical tower stands on a horizontal plane, and is surmounted by a vertical flagstaff of height h. At a point on the plane the angle of elevation of the bottom of the flagstaff is α, and that of the top of the flagstaff is β. Prove that the height of the tower is

$$\frac{h \tan a}{\tan \beta - \tan \alpha}.$$

Evaluate this expression when $h=22$ ft., $\alpha=30° 5'$, $\beta=40°$.

8. P is a point on a horizontal plane through the base A of a vertical mast; Q is a point on the line AP produced, so that P and Q are on the same side of the mast, and AP=100 ft., PQ=30 ft. If the angle of elevation of the top of the mast as seen from P is $75° 50'$, find to the nearest degree the angle of elevation of the top of the mast as seen from Q.

9. The angle of the elevation of the top of a mountain observed from each of three points A, B, C, forming an equilateral triangle of side a on the plane, is α. Show that the height of the mountain is $\dfrac{a}{2} \tan \alpha \operatorname{cosec} A$.

10. Observations to find the height of a mountain are made at two stations A and B which are on a horizontal plane, the distance AB being 4,000 feet; the angle of elevation of the top P of the mountain at A is $60°$, and the angles

PAB and PBA are 75° and 60° respectively. Find the height of the mountain.

11. A vertical tower PQ stands on a hill which is inclined to the vertical at an angle α. At two points A and B, a feet apart on the side of the hill, in the same vertical plane as the tower, the angles subtended by the tower are β and γ. Show that the height of the tower is

$$\frac{a \sin \gamma \sin \beta}{\sin \alpha \sin (\gamma - \beta)}.$$

12. In the above question if $a = 1{,}000$ feet, $\alpha = 30°$, $\beta = 45°$, find the height of the tower.

13. The angle of elevation of the summit of a hill from a station is α; after walking a feet towards the summit up a slope inclined at an angle β to the horizon the angle of elevation is γ. Show that the height of the hill is

$$\frac{a \sin \alpha \sin (\gamma - \beta)}{\sin (\gamma - \alpha)} \text{ feet.}$$

14. The angle of elevation of the top of a tower at a place A due south of it is θ, and at a place B due west of A and at a distance a from it the elevation is ϕ. Show that the height of the tower is

$$\frac{a \tan \theta \tan \phi}{\sqrt{\tan^2\theta - \tan^2\phi}}.$$

15. A tower 51 feet high has a mark at a height of 25 feet from the ground; find at what distance the two parts subtend equal angles at an eye at the height of 5 feet from the ground.

16. At a point on a level plane a tower subtends an angle α, and a man a feet high on its top, an angle β. Prove that the height of the tower is

$$\frac{a \sin \alpha \cos (\alpha + \beta)}{\sin \beta}.$$

17. A tower on the side of a hill is observed to subtend the same angle at two points a feet apart on the same side of it on a horizontal plane. Show that if the angles of elevation of the top of the tower at the two places are α and β

respectively, the height of the tower is $\dfrac{a\cos(\alpha+\beta)}{\sin(\alpha-\beta)}$ feet, where the points are in the same vertical plane as the tower.

18. A flagstaff stands in the middle of a square tower, A man on the ground opposite the middle of a side of the tower, and distant 100 feet from it, just sees the flag; receding another 100 feet, the elevations of the tops of the tower and the flag are found to be α and β respectively, where $\tan\alpha=\frac{1}{2}$, $\tan\beta=\frac{5}{9}$. Find the heights of the tower and the flagstaff.

19. A tower a feet high stands on the top of a cliff b feet high. Find at what point on a horizontal plane passing through the base of the cliff an observer must place himself so that the cliff and the tower may subtend equal angles, the height of the eye being h feet.

If $a=150$ ft., $b=80$ ft. and $h=5$ ft., find the position of the observer.

20. On the bank of a river there is a column 200 feet high supporting a statue 30 feet high. The statue, to an observer on the opposite bank, subtends an equal angle with the man 6 ft. high standing at the base of the column. Find the breadth of the river.

21. A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye and that the angles of depression of their bases are θ_1, θ_2, θ_3; prove that

$$\frac{\sin(\theta_2-\theta_3)}{h_2\sin\theta_1}+\frac{\sin(\theta_3-\theta_1)}{h_2\sin\theta_2}+\frac{\sin(\theta_1-\theta_2)}{h_3\sin\theta_3}=0,$$

h_1, h_2, h_3, being the heights of the towers.

22. At each end of a base of length $2a$ the elevation of the top of a mountain is A, and at the middle point of the base the elevation is B. Prove that the height of the mountain is

$$\frac{a\sin \text{A and B}}{\sqrt{\sin(\text{A}+\text{B})\sin(\text{B}-\text{A})}}.$$

23. An inclined plane rises from the floor of a room at an angle α with the floor and a wheel of radius r is rolled straight up the plane. When the point of contact is at a distance x from the line where the plane meets the floor, show that the highest point of the wheel is at a height above the floor given by the expression $x\sin\alpha+r(1+\cos\alpha)$.

Find the radius of the wheel, if it just touches the ceiling when $x=10$ ft., $\alpha=40°$, and the height of the room is 10 ft.

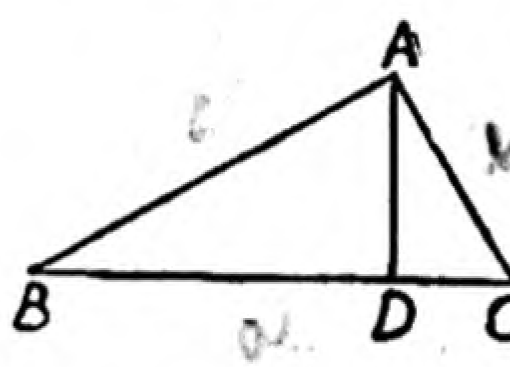
24. From a stationary balloon at a height of 200 ft. an observer notes two points A and B on a st. level road; the bearing of A is due East and that of B is 60° E of N, and their angles of depression are 30° and 45° respectively. Find the distance AB.

CHAPTER XIV

PROPERTIES OF TRIANGLES

98. To find the area of a given triangle ABC.

Draw AD ($=p$, say) $\perp$ BC.



Then area $\triangle$ of the given triangle ABC

$$\Delta = \frac{1}{2} BC.DA \quad \ldots\ldots(1)$$

But $DA = c \sin B$.

$$\left.\begin{array}{l} \therefore\ \Delta = \frac{1}{2}\, ca \sin B \\ \text{Similarly}\ \Delta = \frac{1}{2}\, bc \sin A = \frac{1}{2}\, ab \sin C \end{array}\right\} \quad \ldots\ldots(2)$$

Now $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

Hence
$$\Delta = \frac{1}{2}\, b.\, c.\, \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}. \quad \ldots\ldots(3)$$

Again, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

$$\therefore \quad a = \frac{b \sin A}{\sin B} \text{ and } c = \frac{b \sin C}{\sin B}.$$

Hence
$$\Delta = \frac{1}{2}\, ac \sin B$$
$$= \frac{1}{2} \cdot \frac{b \sin A}{\sin B} \cdot \frac{b \sin C}{\sin B} \cdot \sin B$$

$$\left.\begin{aligned} &= \frac{1}{2}\,\frac{b^2 \sin A \sin C}{\sin B} \\ \text{Similarly}\quad \Delta &= \frac{1}{2}\,\frac{a^2 \sin B \sin C}{\sin A} \\ \text{and also}\quad \Delta &= \frac{1}{2}\,\frac{c^2 \sin A \sin B}{\sin C} \end{aligned}\right\} \qquad \ldots\ldots(4)$$

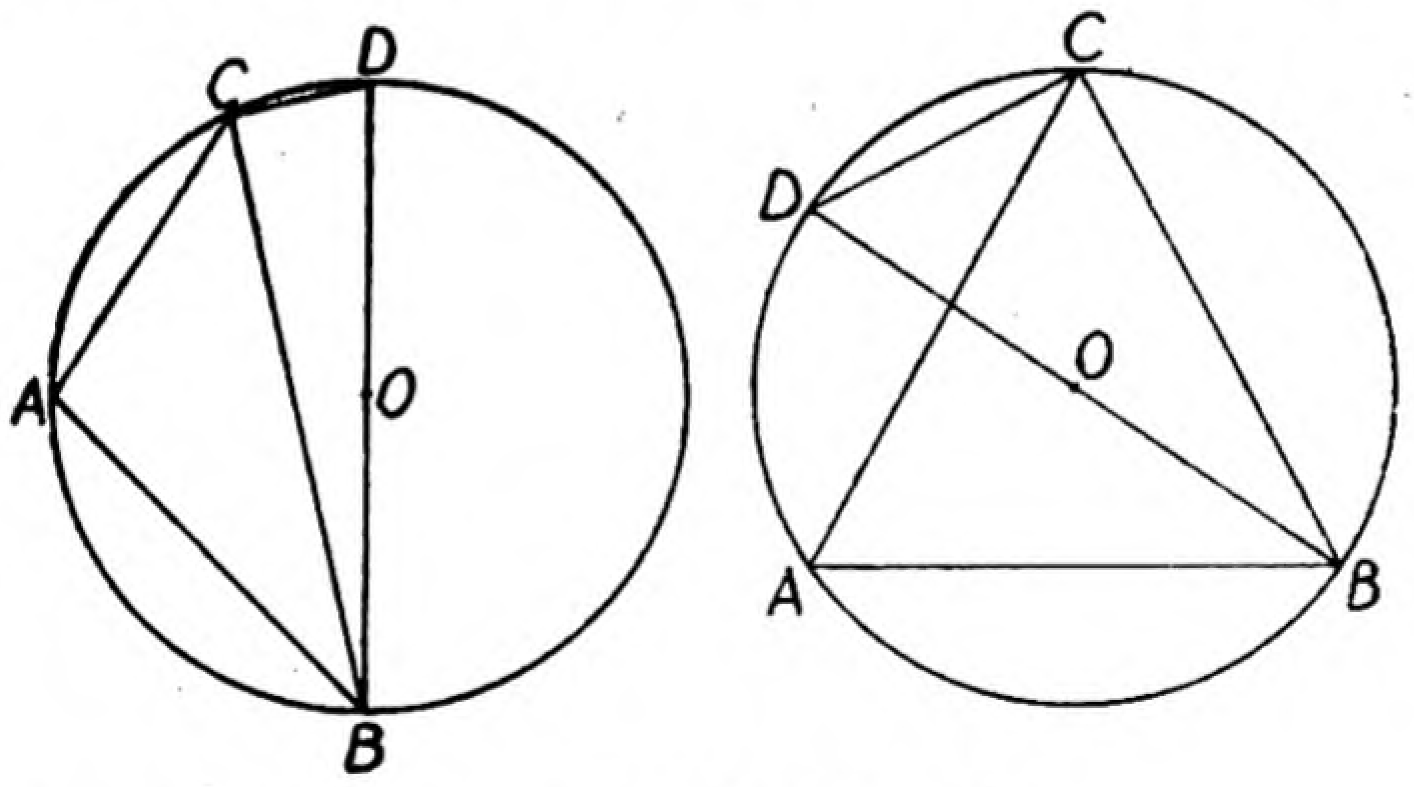
Note.—Form (1) can be used when a side and the corresponding altitude are given; form (2) when two sides and the included angle are given; form (3) when all the three sides are given; and form (4) when one side and two angles are given.

99. To find the circum-radius R of a given triangle ABC.

[**Def.**—The circle through A, B, C, the three vertices of a triangle ABC is called the circumcircle and its centre as the circumcentre. The latter is a point where rt. bisectors of the sides meet.]

Let O be the circumcentre. Join BO and produce it to cut the circle again at D. Join CD; then $\angle BCD$, being in a semi-circle, is a right angle.

Now $\angle BDC = \mathbf{A}$ if A is acute, otherwise $\angle BDC = \pi - A$.



In either case, $\sin \angle BDC = \sin A$.

But $\dfrac{BC}{BD} = \sin \angle BDC$. $\therefore\ \dfrac{a}{2R} = \sin \angle BDC = \sin \mathbf{A}$.

Hence $\mathbf{R} = \dfrac{\mathbf{a}}{\mathbf{2 \sin A}}$.

Similarly $\mathbf{R} = \dfrac{\mathbf{b}}{\mathbf{2 \sin B}} = \dfrac{\mathbf{c}}{\mathbf{2 \sin C}}$.

Another Expression, $R=\dfrac{a}{2\sin A}$

$$=\frac{a}{2\times\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}}=\frac{abc}{4\Delta},$$

giving the radius in terms of the sides.

Note.—It follows that

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$$

so that the above may be regarded as another proof for the sine formulae

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

100. To find the inradius r of a given triangle ABC.

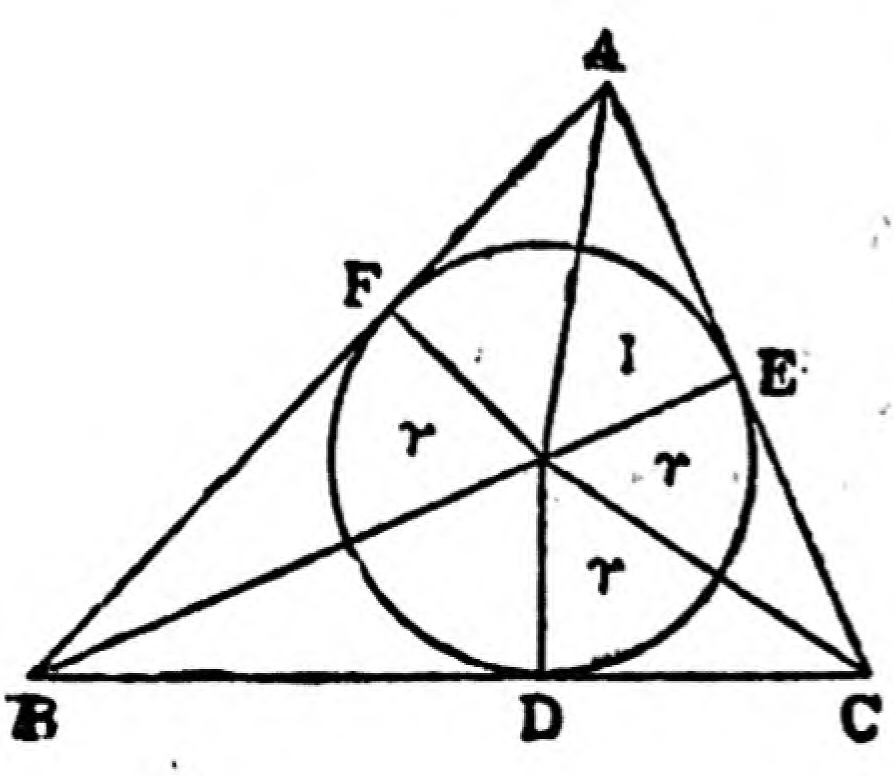
(Def.—If a circle touch all the three sides of a triangle internally, its centre is called the In-centre and it is the point where internal bisectors of the angles meet.)

Draw the internal bisectors of the angles of the given triangle to find the in-centre I. Draw ID, IE, IF, perpendiculars to BC, CA. AB respectively to find the points of contact of the in-circle with the sides. Then $r=ID=IE=IF$.

$\Delta\ ABC=\Delta BIC+\Delta CIA+\Delta AIB$

i.e., $\Delta=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr$

$$=\frac{1}{2}r(a+b+c)=\frac{1}{2}r.\ 2s=sr\quad\therefore\quad r=\frac{\Delta}{s}.$$



Another expression : The two tangents drawn from an external point to a circle are equal. Therefore, AE=AF, CE=CD and BD=BF.

$$\therefore\quad 2s=(AE+AF)+(BF+BD)+(CD+CE)$$

or $$s=AF+BD+DC$$
$$=AF+a$$

or $\quad AF = s - a. \quad \therefore \dfrac{IF}{AF} = \tan\dfrac{A}{2}.$

$$\left.\begin{aligned}
&\text{or} && r = (s-a)\tan\frac{A}{2},\\
&\text{Similarly} && r = (s-b)\tan\frac{B}{2},\\
&\text{and} && r = (s-c)\tan\frac{C}{2}.
\end{aligned}\right\}$$

Ex. 1. Show that

$$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

$$a = BC = BD + DC = \frac{BD}{ID}.ID + \frac{DC}{ID}.ID$$

$$= r\cot\frac{B}{2} + r\cot\frac{C}{2}$$

$$= r\left\{\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right\}$$

$$= r\frac{\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = r\frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\left(\because \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}\right) \quad \therefore r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

Another method :

$$\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = a\sqrt{\frac{(s-c)(s-a)}{ac}} \times \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\times\sqrt{\frac{bc}{s(s-a)}}=\sqrt{\frac{s(s-a)(s-b)(s-c)}{s}}=\frac{\Delta}{s}=r.$$

It may be observed that the same steps could be carried out in the reverse order also.

Ex. 2. Show that $r=4\text{R}\sin\frac{\text{A}}{2}\sin\frac{\text{B}}{2}\sin\frac{\text{C}}{2}$.

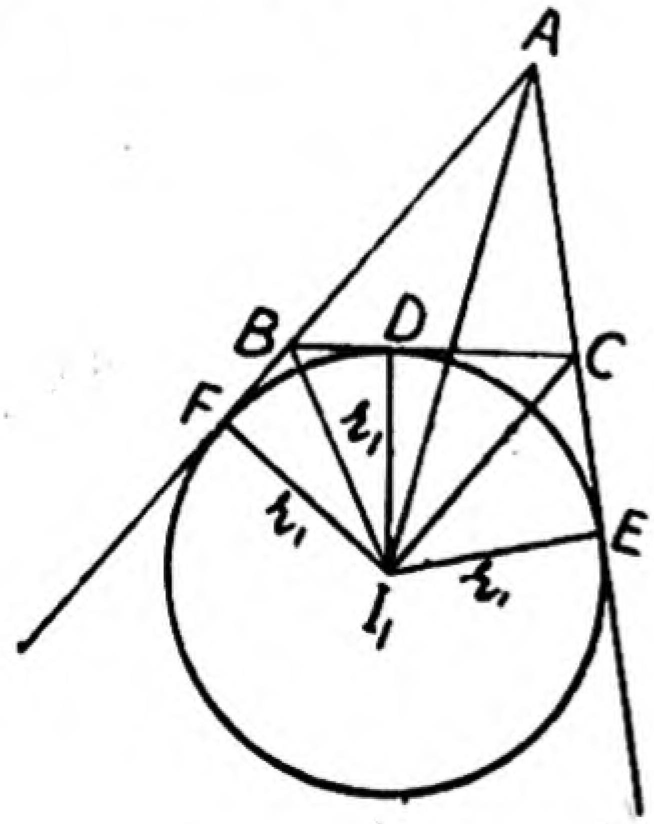
$$4\text{R}\sin\frac{\text{A}}{2}\sin\frac{\text{B}}{2}\sin\frac{\text{C}}{2}=\frac{4abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\times\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{(s-a)(s-b)(s-c)}{\Delta}=\frac{s(s-a)(s-b)(s-c)}{s\Delta}=\frac{\Delta^2}{s\Delta}=\frac{\Delta}{s}=r.$$

$$\therefore\qquad r=4\text{R}\sin\frac{\text{A}}{2}\sin\frac{\text{B}}{2}\sin\frac{\text{C}}{2}.$$

101. To find the radii of the escribed circles of a given triangle ABC.

Draw the internal bisector of the angle A and the external bisectors of the angles B and C to find the ex-centre I_1. Draw I_1D, I_1E and I_1F perpendiculars to BC, CA, AB respectively to find the points of contact of the ex-circle with the sides.



Then $r_1=\text{I}_1\text{D}=\text{I}_1\text{E}=\text{I}_1\text{F}$.

$\Delta\ \text{ABC}=\Delta\ \text{ABI}_1+\Delta\ \text{ACI}_1-\Delta\ \text{BCI}_1$

i.e., $\Delta=\frac{1}{2}cr_1+\frac{1}{2}br_1-\frac{1}{2}ar_1$
$=\frac{1}{2}r_1(c+b-a)$
$=\frac{1}{2}r_1(2s-2a)=r_1(s-a)$

$$\therefore\qquad r_1=\frac{\Delta}{s-a}. \qquad \dots (1)$$

Similarly r_2 (the radius of the circle touching CA and the other two sides produced)$=\frac{\Delta}{s-b}$ and r_3 (the radius of

the circle touching AB and the other two sides produced) $=\dfrac{\Delta}{s-c}$.

Another Expression. The two tangents drawn from an external point to a circle are equal. Therefore BD=BF, CD=DE and AF=AE.

$$\therefore\ 2s=AB+BD+DC+CA=AB+BF+EC+CA$$
$$=AF+AE=2AF.$$

$\therefore$ AF$=s$.

Hence $\dfrac{FI_1}{AF}=\tan\dfrac{A}{2}$ or $\dfrac{r_1}{s}=\tan\dfrac{A}{2}$.

$$\therefore\quad r_1=s\tan\frac{A}{2},$$

Similarly $r_2=s\tan\dfrac{B}{2}$,

and $r_3=s\tan\dfrac{C}{2}$.

Again $r_1=(s-b)\cot\dfrac{C}{2}$

$=(s-c)\cot\dfrac{B}{2}$

$r_2=(s-a)\cot\dfrac{C}{2}$

$=(s-c)\cot\dfrac{A}{2}$

$r_3=(s-a)\cot\dfrac{B}{2}$

$=(s-b)\cot\dfrac{A}{2}$.

Ex. 1. Show that

$$r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}},\ r_2=\frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$$

and $$r_3=\frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}.$$

$$a=BD=BC+DC=\frac{BD}{I_1D}\cdot I_1D+\frac{DC}{I_1D}\cdot I_1D$$
$$=I_1D\cot I_1BD+I_1D\cot I_1CD$$

$$=r_1 \tan\frac{B}{2}+r_1\tan\frac{C}{2}$$

$$=r_1\left\{\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right\}$$

$$=r_1\,\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=r_1\,\frac{\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=r_1\,\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$\therefore\quad r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

Similarly the other expressions follow.

Another Method :

$$\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}=a\sqrt{\frac{s(s-b)}{ca}}.\sqrt{\frac{s(s-c)}{ab}}.\sqrt{\frac{bc}{s(s-a)}}$$

$$=\sqrt{\frac{s(s-b)(s-c)}{s-a}}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}=\frac{\Delta}{s-a}=r_1.$$

It may be observed that the same steps could be carried out in the reverse order also.

Ex. 2. Show that $r_1=4\,R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

$$4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=\frac{4abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{s(s-b((s-c)}{\triangle}=\frac{s(s-a)(s-b)(s-c)}{(s-a)\triangle}=\frac{\triangle}{s-a}=r_1.$$

$\therefore \quad r_1=4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

Similarly $r_2=4R\sin\frac{B}{}\cos\frac{C}{2}\cos\frac{A}{2}.$

and $r_3=4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$

Ex. 3. Show that $\triangle=r\,r_1\cot\frac{A}{2}.$

$$r\,r_1\cot\frac{A}{2}=\frac{\triangle}{s}\cdot\frac{\triangle}{s-a}\cdot\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\frac{\triangle.\triangle}{\sqrt{s(s-a)(s-b)(s-c)}}=\triangle.$$

Ex. 4. Show that $r_1r_2+r_2r_3+r_3r_1=s^2$. (B.U.)

$$\text{L. H. S.}=\frac{\triangle}{s-a}\cdot\frac{\triangle}{s-b}+\frac{\triangle}{s-b}\cdot\frac{\triangle}{s-c}+\frac{\triangle}{s-c}\cdot\frac{\triangle}{s-a}$$
$$=s(s-c)+s(s-a)+s(s-b)$$
$$=s(3s-a-b-c)=s^2.$$

Ex. 5. Given r_1, r_2 and r_3 ; find A, B, C.

$r_1=s\tan\frac{A}{2}$, $r_2=s\tan\frac{B}{2}$, $r_3=s\tan\frac{C}{2}.$

Therefore $\tan\frac{B}{2}=\frac{r_2}{r_1}\tan\frac{A}{2}$, $\tan\frac{C}{2}=\frac{r_3}{r_1}\tan\frac{A}{2}.$

Also, we know that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

Substituting the values of $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$, we get

$$\tan^2\frac{A}{2}\left(\frac{r_2}{r_1}+\frac{r_2r_3}{r_1^2}+\frac{r_3}{r_1}\right)=1,$$

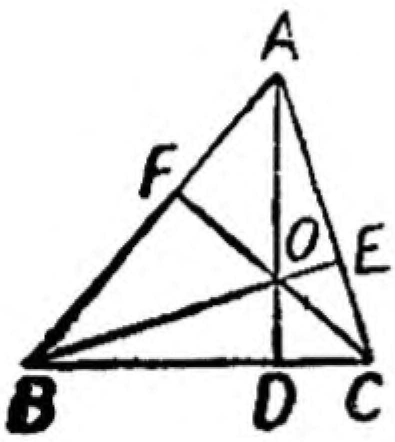
or $\tan\frac{A}{2}=\frac{r_1}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}$ because $\frac{A}{2}$ is acute.

[Otherwise this result follows at once if the result of Ex. 4 solved above be assumed.]

102. Miscellaneous solved Examples.

Ex. 1. *Find the distance of the ortho-centre from (i) the sides (ii) the angular points of a triangle.*

Let ABC be the triangle. Draw AD, BE perpendiculars to BC and AC and let O be the ortho-centre,



Then $OD = BD \tan \angle OBD$

$= BD \tan (90° - C)$

$= BD \cot C.$

Also $BD = AB \cos ABD$

$= c \cos B.$

$OD = c \cos B \cot C$

$= \dfrac{c}{\sin C} \cos B \cos C$

$= 2R \cos B \cos C.$

Thus distances of O from the sides BC, CA, AB are respectively.

$2R \cos C \cos B,\ 2R \cos A \cos C,\ 2R \cos B \cos A.$

Now $AO = OE \operatorname{cosec} \angle OAE$

$= OE \operatorname{cosec} (90° - C)$

$= 2R \cos A \cos C \sec C$

$= 2R \cos A,$

$\therefore$ The required distances from the angular points are
$2R \cos A,\ 2R \cos B,\ 2R \cos C.$

Ex. 2. *Perpendiculars ID, IE, IF are drawn from the incentre I of the triangle ABC on the sides BC, CA, AB respectively. Show that the area of the triangle* $DEF = \dfrac{r\Delta}{2R}$ *where r, R and Δ have got their usual meanings.*

Here $\Delta\, DEF$

$= \Delta\, IED + \Delta\, IDF + \Delta\, IFE$

$= \tfrac{1}{2} . IE . ID . \sin EID +$ two similar terms

$= \tfrac{1}{2}r^2 \sin (180° - C) + \tfrac{1}{2}r^2 \sin (180° - B) + \tfrac{1}{2}r^2 \sin (180° - A)$

$= \tfrac{1}{2}r^2 [\sin C + \sin B + \sin A]$

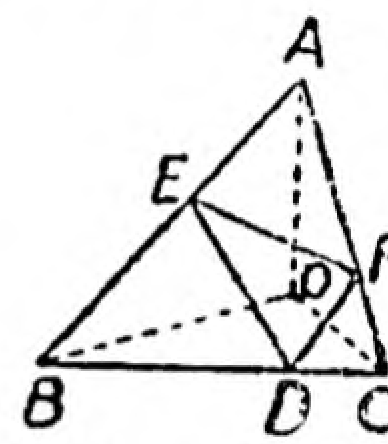
$= \tfrac{1}{2}r^2 \left[\dfrac{a}{2R} + \dfrac{b}{2R} + \dfrac{c}{2R}\right]$

$$=\frac{r^2\times 2s}{4R}=\frac{r\times\triangle\times 2s}{4R\times s}=\frac{r\triangle}{2R}.$$

Ex. 3. *Find the sides and angles of the pedal triangle of a given triangle.*

[**Def.**—The triangle formed by the three feet of the perpendicular drawn from the vertices on the opposite sides of the triangle is called pedal triangle of the original triangle.]

Draw AD, BF, CE perpendiculars to sides BC, CA, AB respectively to meet at O. Then O is the orthocentre.



Here $\angle EDF=\angle EDO+\angle FDO$

$=\angle EBO+\angle FCO$......[?]

$=\angle ABF+\angle ACE$

$=90°-A+90°-A=180°-2A.$

Similarly $\angle DEF=180°-2C$ and $\angle EFD=180°-2B$.

Again by sine formulæ for $\triangle FCD$ we have

$$\frac{DF}{\sin C}=\frac{CD}{\sin B}$$

$$\therefore\quad DF=\frac{CD\sin C}{\sin B}=\frac{b\cos C\sin C}{\sin B}$$

$$=2R\cos C\sin C=c\cos C.$$

EXERCISE XXXVII

1 The sides of a triangle are 8 ft. and 7 ft. and the included angle is 60° . find the area.

2. Find the area of a triangle whose sides are 7 ft. 9ft. and 8 ft.

3. Calculate the area of a triangle if $b=4{\cdot}35$ inches, $c=7{\cdot}55$ inches and $A=152°$,

4. The area of a triangle is 135 square ft. and the lengths of the ex-radii are 3 ft., 5ft., and 9 ft., respectively. Find the sides of the triangle.

5. The sides of a triangle are 4, 13, 15, feet ; find the greatest angle and the least altitude of the triangle.

6. Two sides of a triangle are 1 ft. and $\sqrt{2}$ ft., and the

angle opposite to the shorter side is 30° ; prove that there are two triangles satisfying these conditions, whose areas are in the ratio of $\sqrt{3}+1 : \sqrt{3}-1$.

7. If the sides of a triangle are 51, 68 and 85 ft., show that the shortest sides is divided by the point of contact of the inscribed circle into two segments, one of which is double the other.

Prove that

8. $s(s-a)\tan\frac{A}{2}=\Delta.$ 9. $\frac{a^2-b^2}{2}\cdot\frac{\sin A\sin B}{\sin(A-B)}=\Delta.$

10. $b^2\sin 2C+c^2\sin 2B=4\Delta.$

11. $a^2-b^2=2Rc\sin(A-B).$

12. $4R\cos\frac{C}{2}=(a+b)\sec\frac{A-B}{2}.$

13. $4R=s\sec\frac{A}{2}\sec\frac{B}{2}.\sec\frac{C}{2}.$

14. $\Delta=2R^2\sin A\sin B\sin C.$

15. In any triangle ABC, join C to any point D in AB ; let R, R_1 be circumradii of the triangles ACD and BCD respectively ; show that $Ra=R_1b$.

16. $r=\dfrac{a-b}{\cot\frac{B}{2}-\cot\frac{A}{2}}.$ 700

17. $r\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)=s.$

18. $\frac{r}{4R}=\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right).$

19. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}.$ (A. U.)

20. $\Delta=Rr(\sin A+\sin B+\sin C).$

21. $abc.s.\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\Delta^2.$

22. $Rr_1(s-a)=Rr_2(s-b)=Rr_3(s-c)=\frac{1}{4}abc.$

23. $rr_1r_2r_3=\Delta^2.$

24. $rr_1=r_2r_3\tan^2\frac{A}{2}.$ 25. $rr_1\cot\frac{A}{2}=\Delta.$

26. $\Delta = 4Rr \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$.

27. $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$.

28. If the ex-circle opposite to the angle A be equal to the circumcircle, prove that

$$\cos A = \cos B + \cos C.$$

29. Show that $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$.

30. If Δ and Δ' are the areas of a triangle ABC and the triangle formed by joining the points of contact of its inscribed circle, prove that

$$\frac{\Delta'}{\Delta} = \frac{2(s-a)(s-b)(s-c)}{abc}.$$

31. Prove that the ratio of the area of the inscribed circle of a triangle ABC to the area of the triangle is

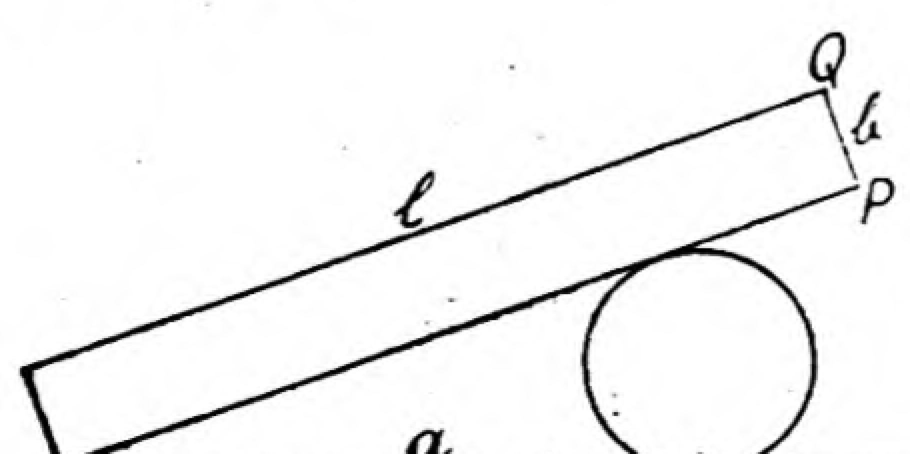
$$\pi \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}.$$

32. AB is a chord of a circle of radius R and subtends an angle 2α at the centre C. Prove that the radius of the circle inscribed in the triangle ABC is $R \tan\alpha(1-\sin\alpha)$.

33. Show that $\frac{r_2+r_3}{(s-a)\sin A} = \frac{r_3+r_1}{(s-b)\sin B}$

$$= \frac{r_1+r_2}{(s-c)\sin C}. \qquad \text{(B. U.)}$$

34. A baulk of wood of the dimensions shown in the figure rests over a cylinder of radius r. Find the heights of P and Q above the ground.



35. The vertical angle of an isosceles triangle is A, and the length of each of the equal sides is b; prove that the radius of the inscribed circle is

$$\frac{b \sin \frac{A}{2}}{\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)}.$$

Formulae of Chapter XIV

1. $\Delta=\frac{1}{2} bc \sin A=\frac{1}{2} ca \sin B=\frac{1}{2} ab \sin C$
$=\sqrt{s(s-a)(s-b)(s-c)}.$

2. $R=\dfrac{a}{2 \sin A}=\text{etc.}$
$=\dfrac{abc}{4\Delta}.$

3. $r=\dfrac{\Delta}{s}$
$=(s-a)\tan\dfrac{A}{2}=\text{etc.}$

4. $r_1=\dfrac{\Delta}{s-a}$
$=s \tan \dfrac{A}{2}$
$=(s-b) \cot \dfrac{C}{2}$
$=(s-c) \cot\dfrac{B}{2}$. Similarly for r_2 and r_3.

REVISION QUESTIONS XI

1. Prove that the area of a triangle is equal to
$\dfrac{b^2+c^2-a^2}{4 \cot A}$.

2. $\Delta=\dfrac{a^2-b^2}{2} \cdot \dfrac{\sin A \sin B}{\sin (A-B)}$.

3. Prove that the radius of the circle inscribed in the pedal triangle of a triangle ABC is 2R cos A cos B cos C, where R is the circumradius of the triangle ABC.

4. The perpendiculars from the circumcentre of a triangle on its three sides are α, β, γ, respectively; show that

$$4\left(\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right)=\frac{abc}{\alpha\beta\gamma}.$$

5. If A be the area of the circle inscribed in a triangle and A_1, A_2, A_3 be the areas of the escribed circles, prove that

$$\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}.$$

Prove that

6. $\cos A+\cos B+\cos C=1+\frac{r}{R}$. (B. U. *and* P. U.)

7. $\sin A+\sin B+\sin C=\frac{s}{R}$. (B. U.)

8. The distance between the orthocentre and the circumcentre is $R\sqrt{(1-8\cos A\cos B\cos C)}$.

9. $\tan^2\frac{A}{2}=\frac{rr_1}{r_2r_3}$.

10. $a\cot A+b\cot B+c\cot C=2R+2r$.

11. $a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C$.

12. Prove that if C be a right angle, $2(R+r)=(a+b)$.

13. $r^2+r_1^2+r_2^2+r_3^2=16R^2-a^2-b^2-c^2$.

14. Find the distance of orthocentre of $\triangle$ ABC from the side AB.

15. Find the distance of the circumcentre of a $\triangle$ ABC from the side AB.

16. Find the distance of the centroid of $\triangle$ ABC from the sides.

17. The line joining the vertex of a triangle ABC to its circumcentre meets BC in D. Show that

$$\frac{BD}{DC}=\frac{\sin 2C}{\sin 2B}.$$

18. Prove that the sides of a triangle whose vertices are the centres of e-circles are $a/\sin\frac{A}{2}$, $b/\sin\frac{B}{2}$, $c/\sin\frac{C}{2}$.

19. Show that $\triangle IBC=8R\sin\frac{A}{2}\sin\frac{B}{4}\sin\frac{C}{4}\cos\frac{B+C}{4}$.

(B. U.)

20. Show that

(*i*) $OI^2=R^2-2Rr$

(*ii*) $OI_1^2=R^2+2Rr_1$

21. Show that if the in-circle of triangle passes through the circumcentre then $\cos A+\cos B+\cos C=\sqrt{2}$.

22. If the line joining the vertex A of a triangle ABC to the centre of the inscribed circle meets the opposite side in D prove that

$$\tan ADB=\frac{b+c}{b-c}\tan\frac{A}{2}.$$

23. In any triangle ABC prove that $p\cos A+q\cos B+r\cos C=\frac{a^2+b^2+c^2}{4R}$, p, q, r being the lengths of the perpendiculars from the vertices upon the opposite sides.

CHAPTER XV

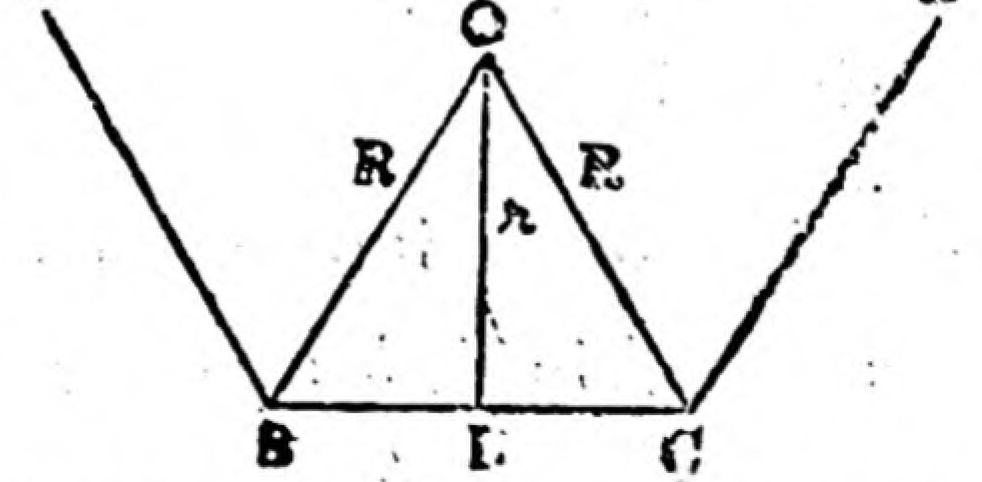
POLYGONS : AREA OF A CIRCLE

103. *A polygon is said to be regular when all its sides are equal and its angles are equal.*

Let a regular polygon have n sides. Then the sum of interior angles $+4$ right angles$=2n$ right angles; *i.e.*, if one interior angle be denoted by x, then $nx=2n-4$ right angles; or $x=\frac{2n-}{n}$ right angles.

104. To find the radii of the inscribed and circumscribed circles of a regular polygon of n sides.

Let ABC .. be a regular polygon of n sides. Let the bisectors of the angles B and C meet in O.



Draw $OL\perp BC$. Then O is the centre of both the circles; OB $(=R$, say) is the radius of the circumscribed circle, and OL $(=r$, say) is the radius of the inscribed circle.

If O be joined to each of the angular points A, B, C, D, .. , the whole angle at O will be divided into n equal parts

and hence $\angle BOC = \frac{2\pi}{n}$ and $\angle BOL = \angle COL = \frac{\pi}{n}$.

Let each side of the polygon be equal to a.

Then $\frac{BL}{OB} = \sin \angle BOL$, *i.e.*, $\frac{a}{2R} = \sin \frac{\pi}{n}$.

$$\therefore \quad R = \frac{a}{2 \sin \frac{\pi}{n}}.$$

Also $\frac{BL}{OL} = \tan BOL$, *i.e.*, $\frac{a}{2r} = \tan \frac{\pi}{n}$;

$$\therefore \qquad r = \frac{a}{2 \tan \frac{\pi}{n}}.$$

105. To find the area of a regular polygon of n sides.

Area of the whole regular polygon of n sides $= n$ times the area of the triangle BOC

$$= n \times \tfrac{1}{2} \times OB.OC \sin BOC$$

$$= \frac{n}{2} R^2 \sin \frac{2\pi}{n} \text{ (in terms of R).}$$

Also area of the polygon $= n. \tfrac{1}{2}$ OL.BC

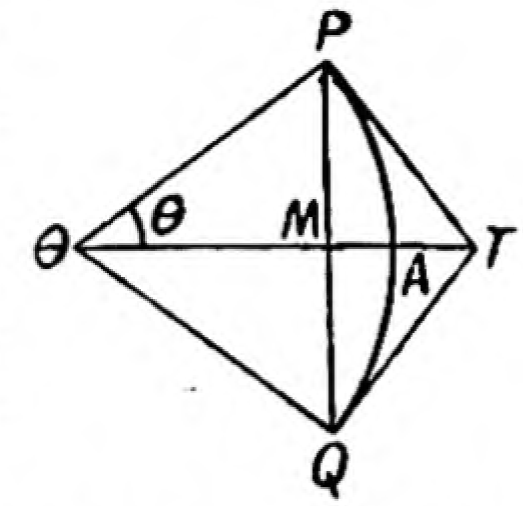
$$= \frac{na}{2} \times \frac{a}{2} \cot \frac{\pi}{n}$$

$$= \frac{na^2}{4} \cot \frac{\pi}{n} \text{ (in terms of } a\text{)}$$

Again, area of polygon $= \frac{n}{2}$ OL.BC $= nr. \tan \frac{\pi}{n}$

$$= nr^2 \tan \frac{\pi}{n} \text{ (in terms of } r\text{).}$$

106. If θ be the circular measure of any angle which is less then a right angle, then $\sin \theta$, θ and $\tan \theta$ are in ascending order of magnitude.



Let TOP be an angle which is less than a right angle and let its circular measure be θ.

With O as centre and any radius describe an arc cutting OT and OP at A and P respectively; also draw PM $\perp$ OA and produce it to cut the arc again in Q.

Draw the tangent PT at P to meet OA in T, and join TQ and OQ.

The triangles POM and QOM are evidently equal in all respects; so that

$$MP=MQ \text{ and arc } PA=\text{arc } AQ.$$

Also the triangles TOP and TOQ are equal in all respects, so that TP=TQ.

The straight line PC is less than the arc PAQ, so that MP is less than arc AP.

Also we shall *assume* that the arc PAQ is less than PT+TQ, so that arc PA is less than PT.

Hence MP, arc AP and PT are in ascending order of magnitude. Therefore

$\frac{MP}{OP}, \frac{\text{arc } AP}{OP}, \frac{PT}{OP}$ are in ascending order of magnitude.

Hence $\sin\theta$, θ and $\tan\theta$ arc in ascending order of magnitude.

107. Show that $\underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta}=1$, provided that the angle θ is measured in radians.

Since $\sin\theta<\theta<\tan\theta$ when $\theta<\frac{\pi}{2}$

$$\therefore \quad 1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}$$

i.e., $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$.

But when θ approaches zero, $\cos\theta$ approaches unity and consequently $\frac{\theta}{\sin\theta}$, which lies between 1 and $\frac{1}{\cos\theta}$ also approaches unity.

Therefore $\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\sin\theta}{\theta}=1.$

Cor. 1. $\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\tan\theta}{\theta}=\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}=1.$

Cor. 2. $\underset{n\to\infty}{\mathrm{Lt}}\ \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}=1.$

Ex. 1. Show that $\underset{n\to\infty}{\mathrm{Lt}}\ n\sin\frac{\theta}{n}=\theta$, when θ is measured in radians.

$$\underset{n\to\infty}{\mathrm{Lt}}\ n\sin\frac{\theta}{n}=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\theta=\theta.$$

Ex. 2. Find $\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sin x^\circ}{x}$.

Now $180^\circ=\pi$ radians.

$\therefore\ x^\circ=\frac{\pi}{180}x$ radians.

Hence $\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sin x^\circ}{x}=\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sin\frac{\pi x}{180}}{x}=\underset{x\to 0}{\mathrm{Lt}}\ \frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\cdot\frac{\pi}{180}$

$$=\frac{\pi}{180}.$$

Ex. 3. Euler's Theorem—Show that

$$\sin\theta=\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cdots \qquad \text{(P. U. 1940)}$$

Here $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. Again since $\sin\frac{\theta}{2}=2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}$

$$\therefore \quad \sin\theta = 2^2 \sin\frac{\theta}{2^2} \cos\frac{\theta}{2} \cos\frac{\theta}{2^2}$$

$$= 2^3 \sin\frac{\theta}{2^3} \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3}$$

$$= 2^n \sin\frac{\theta}{2^n} \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3} \ldots\ldots \cos\frac{\theta}{2^n}$$

$$= \frac{\sin\dfrac{\theta}{2^n}}{\dfrac{1}{2^n}} \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3} \ldots\ldots \cos\frac{\theta}{2^n}.$$

Now when $n \to \infty$, $2^n \to \infty$ and $\therefore \quad \dfrac{\theta}{2^n} \longrightarrow 0$.

Thus, $$\frac{\sin\dfrac{\theta}{2^n}}{\dfrac{1}{2^n}} = \theta \, \frac{\sin\dfrac{\theta}{2^n}}{\dfrac{\theta}{2^n}} \to \theta$$

$$\therefore \quad \sin\theta = \theta \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3} \ldots\ldots$$

108. (*a*) **Area of a circle.** The area of a regular polygon of n sides in terms of the radius of the circumscribed circle is equal to $\dfrac{n}{2} R^2 \sin\dfrac{2\pi}{n}$.

Now let the number of sides of this polygon be infinitely increased, the polygon always remaining regular.

It is clear that the perimeter of the polygon must more and more coincide with circumference.

Hence the area of the circle

$$= \underset{n \to \infty}{\mathrm{Lt}} R^2 \sin\frac{2\pi}{n},$$

$$= \underset{n \to \infty}{\mathrm{Lt}} \frac{n}{2} R^2 \frac{2\pi}{n} \frac{\sin\dfrac{2\pi}{n}}{\dfrac{2\pi}{n}} = \underset{n \to \infty}{\mathrm{Lt}} \pi R^2 \frac{\sin\dfrac{2\pi}{n}}{\dfrac{2\pi}{n}} + \pi R^2.$$

(*b*) **Circumference of a circle.** The circumference of a

regular polygon of n sides in terms of the radius of the circumscribed circle is equal to

$$n\,2R\sin\frac{\pi}{n}.$$

Now let the number of sides of this polygon be infinitely increased, the polygon always remaining regular. The perimeter of the polygon must more and more coincide with the circumference.

Hence the circumference of the circle

$$=\underset{n\to\infty}{\mathrm{Lt}}\, n.2R\sin\frac{\pi}{n}=\underset{n\to\infty}{\mathrm{Lt}}\, n2R\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\frac{\pi}{n}$$

$$\underset{n\to\infty}{\mathrm{Lt}}\, 2\pi R\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}=2\pi R.$$

109. *Area of a sector of a circle.*

Let AB be an arc of a circle, centre O and radius R and let $\angle AOB=\alpha$ radians

$$\frac{\text{Area of the sector AOB}}{\text{Area of the circle}}=\frac{\angle AOB}{2\pi}=\frac{\alpha}{2\pi}\cdot$$

Hence the area of the sector AOB

$$=\frac{\alpha}{2\pi}\pi R^2=\frac{\alpha R^2}{2}.$$

Cor. Since $\alpha=\dfrac{\text{arc AB}}{R}$, area of the sector is also equal to $\dfrac{\text{Arc AB}\times R}{2}$.

Thus area of a sector$=\dfrac{\text{Arc}\times\text{Radius}}{2}$.

EXERCISE XXXVIII

1. If R and r be the radii of the circumcircle and the incircle, of a regular polygon of n sides, each equal to a,

prove that

$$R+r=\frac{a}{2}\cot\frac{\pi}{2n}.$$

2. Prove that the perimeters of the circumscribing polygon, the circle and the inscribed polygon are in the ratio

$$\sec\frac{\pi}{n}:\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}:1,$$

and that the areas of the polygons are in the ratio

$$\cos^2\frac{\pi}{n}:1.$$

3. The area of a polygon of n sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as $3:2$; show that $n=3$.

4. If a triangle be formed with sides of the regular hexagon, pentagon and decagon inscribed in the same circle, the triangle is right-angled. (P. U. 1938).

5. If R, r be the radii of the circumscribed and inscribed circles of a regular polygon and R′ and r' those of the regular polygon of the same area but double the number of sides, show that

$$R'=\sqrt{Rr}\text{ and }r'=\sqrt{\frac{r}{2}(R+r)}.$$

6. A polygon has circles of radii R and r described about and inscribed in it. A new polygon of which the radius of the inscribed circle is r', is formed by joining the points of contact of the original polygon with its inscribed circle; prove that $r^2=Rr'$.

7. A polygon of $2n$ sides of which n are equal to a and n equal to b, is inscribed in a circle, show that radius of the circle is

$$\frac{1}{2}\left(a^2+2ab\cos\frac{\pi}{n}+b^2\right)^{\frac{1}{2}}\operatorname{cosec}\frac{\pi}{n}.$$

8. Illustrate with the help of tables that $\frac{\sin\theta}{\theta}$ approaches unity as the angle becomes smaller and smaller, by taking θ equal to the circular measure of 4°, 3°, 2°, 1° and 30′.

9. Prove that $\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1+\theta}{\sqrt{2}}$ approximately when θ is small.

10. Show that the area of a segment of a circle is given by the formula $\frac{1}{2}r^2(\theta-\sin\theta)$.

11. Four equal circles each of radius a touches one another; show that the area between them is $a^2(4-\pi)$.

12. Three equal circles of radius a touch one another: show that the area between them is $\left(\sqrt{3}-\frac{\pi}{2}\right)a^2$.

13. Given the three sides $a=58{\cdot}6$, $b=64{\cdot}3$, and $c=52{\cdot}5$, calculate the area of the inscribed circle. (P .U. 1945).

CHAPTER XVI

MISCELLANEOUS PROPOSITIONS

110. *To find the area of a quadrilateral in terms of the sides and the sum of two opposite angles.*

Let ABCD, be the quadrilateral and let a, b, c, and d be the lengths of its sides and S the area.

Equating the two values of BD^2 from the triangles BAD and BCD, we have

$$a^2+d^2-2ad\cos A=b^2+c^2-2bc\cos C$$

$$\therefore\quad a^2+d^2-b^2-c^2=2ad\cos A-2bc\cos C. \qquad \ldots\ldots(i)$$

Also $S=\triangle BAD+\triangle BCD$

$$=\tfrac{1}{2}ad\sin A+\tfrac{1}{2}bc\sin C$$

$$\therefore\quad 4S=2ad\sin A+2bc\sin C. \qquad \ldots\ldots(ii)$$

Squaring and adding (i) and (ii), we have

$$16S^2+(a^2+d^2-b^2-c^2)^2=4a^2d^2+4b^2c^2-8abcd\cos(A+C).$$

Let $A+C=2\beta$, so that

$$\cos(A+C)=\cos 2\beta=2\cos^2\beta-1;$$

$$\therefore\quad 16S^2=4(ad+bc)^2-(a^2+d^2-b^2-c^2)^2-16abcd\cos^2\beta.$$

Now the first two terms on the right-hand side

$$=(2ad+2bc+a^2+d^2-b^2-c^2)(2ad+2bc-a^2-d^2+b^2+c^2)$$

$$=\{(a+d)^2-(b-c)^2\}\{(b+c)^2-(a-d)^2\}$$

$$=(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d)$$

$$=(2s-2c)(2s-2b)(2s-2d)(2s-2a),$$

where $a+b+c+d=2s$,

$$=16(s-a)(s-b)(s-c)(s-d).$$

Hence

$S=\sqrt{\{(s-a)(s-b)(s-c)(s-d)-abcd\ \cos^2\beta\}}$.

Cor. 1. If the quadrilateral is cyclic, then

$S=\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$,

for $A+C=180°$ and $\therefore$ $\cos\beta=\cos 90°=0$.

Cor. 2. *If the quadrilateral is a cyclic and also a pericyclic one, i.e., a circle can be inscribed in it, then*

$$S=\sqrt{abcd}.$$

For, if a circle can be inscribed in a quadrilateral ABCD then the sum of one pair of the opposite sides is equal to that of the other pair, $\therefore$ $a+c=b+d$ and the above expression $\frac{1}{16}(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d)$ for S^2 reduces to $abcd$.

111. Prove Geometrically

(*i*) $\sin P+\sin Q=2\sin\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$

(*ii*) $\sin P-\sin Q=2\cos\dfrac{P+Q}{2}\sin\dfrac{P-Q}{2}$

(*iii*) $\cos P+\cos Q=2\cos\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$

(*iv*) $\cos Q-\cos P=2\sin\dfrac{P+Q}{2}\sin\dfrac{P-Q}{2}$.

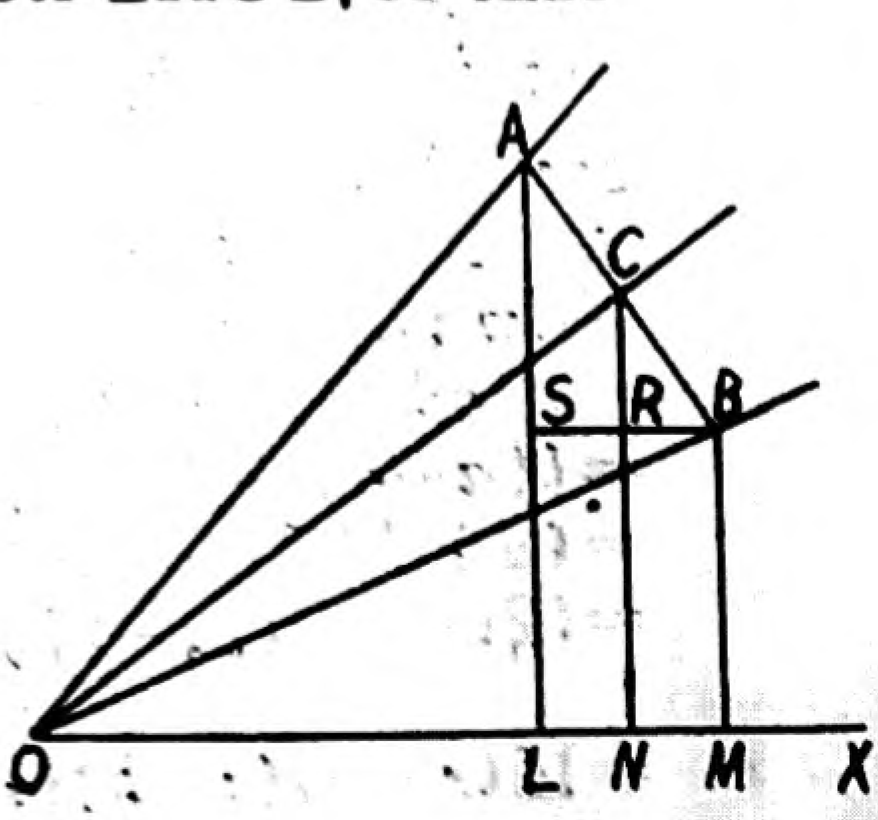
Let $\angle XOA=P$ and $\angle XOB=Q$. Cut off $OA=OB=a$, say. Join AB and bisect it at C. Draw AL, BM, CN perpendiculars to OX. Then OC bisects $\angle AOB$, so that

$\angle COB=\dfrac{1}{2}(P-Q)$ and

$\angle COX=\angle COB+\angle BOX$

$=\dfrac{1}{2}(P-Q)+Q$

$=\dfrac{1}{2}(P+Q)$.



In the trapezium ALMB,

$AL+BM=2CN.$ ......(i)

Also $OL+OM=2ON.$ ......(ii)

Now $OC=OB\cos COB=a\cos\frac{P-Q}{2}$ and

$$ON=OC\cos COX=a\cos\frac{P-Q}{2}\cos\frac{P+Q}{2}$$

$$OL=OA\cos AOX=a\cos P \text{ and}$$
$$OM=OB\cos BOX=a\cos Q.$$

Substituting in (ii) we get

$$a\cos P+a\cos Q=2a\cos\frac{P-Q}{2}\cos\frac{P+Q}{2}$$

i.e., $$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$

Again, $AL=OA\sin AOX=a\sin P,$
$BM=OB\sin BOX=a\sin Q,$

and $CN=OC\sin COX=a\cos\frac{P-Q}{2}\sin\frac{P+Q}{2},$

Substituting in (i), we get

$$a\sin P+a\sin Q=2a\cos\frac{P-Q}{2}\cos\frac{P+Q}{2},$$

i.e., $$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$

Again draw BS ∥ OX cutting CN; AL in R and S.

$AL-BM=AS=2CR,$ ......(iii)

and $OM-OL=LM=2BR.$ ... (iv)

$$\angle BCR=90°-\angle OCR=\angle COX=\tfrac{1}{2}(P+Q),$$

and $BC=OB\sin BOC=a\sin\frac{P-Q}{2},$

and $CR=BC\cos BCR=a\sin\frac{P-Q}{2}\cos\frac{P+Q}{2}.$

Substituting in (iii), we get

$$a\sin P-a\sin Q=2a\sin\frac{P-Q}{2}\cos\frac{P+Q}{2},$$

i.e., $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$.

$$BR = BC\sin BCR = a\sin\frac{P-Q}{2}\sin\frac{P+Q}{2}.$$

Substituting in (*iv*) we get

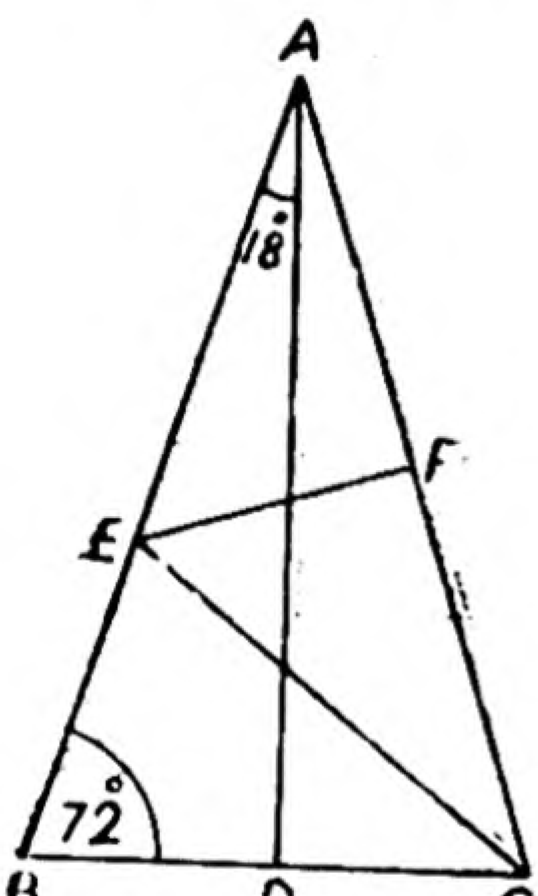
$$a\cos Q - a\cos P = 2a\sin\frac{P-Q}{2}\sin\frac{P+Q}{2}$$

i.e., $\cos Q - \cos P = 2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$.

112. The values of the circular functions of 18° and 36° may be obtained geometrically from the construction for drawing an isosceles triangle ABC in which each angle at the base BC is double the vertical angle A.

$$A+B+C=180^\circ$$

or $A+2A+2A=180^\circ$



$\therefore \quad A=36^\circ$

Draw $AD \perp BC$, so that $\angle BAD = 18^\circ$.

In the construction of the triangle we have

$$BC=AE \text{ and } BE.\,BA=EA^2.$$

Then $\sin 18^\circ = \frac{BD}{AB} = \frac{x}{c}$,

where $BD = x$

$AE = BC = 2x$

and $AB.EB = AE^2$

$\therefore \quad c(c-2x) = (2x)^2$

or $4x^2+2cx-c^2=0$.

$$\therefore \quad x = \frac{-2c \pm \sqrt{20c^2}}{8} = \frac{-1\pm\sqrt{5}}{4}c$$

$= \frac{-1+\sqrt{5}}{4}c$ (rejecting the negative value of x because x is positive).

Hence $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$.

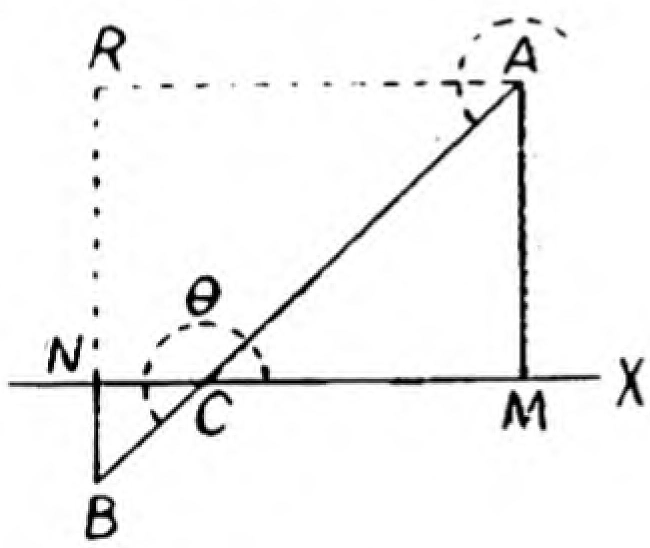
From the same figure we can find the value of cos 36° as well. Draw EF⊥AC. Also AE=EC. Hence F bisects AC;

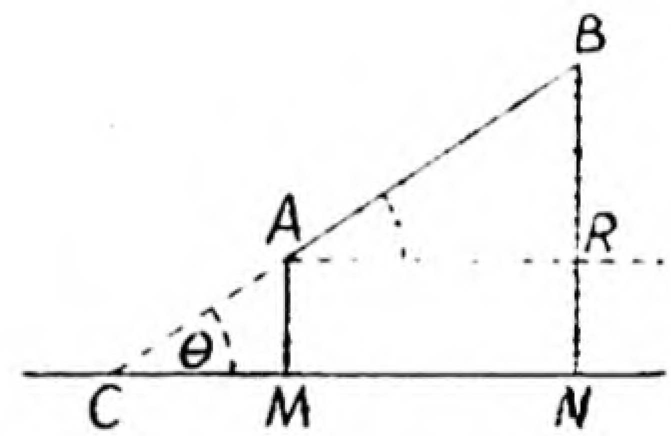
therefore $AF=\frac{1}{2}\ AC=\ \frac{1}{2}\ AB=\frac{c}{2}$.

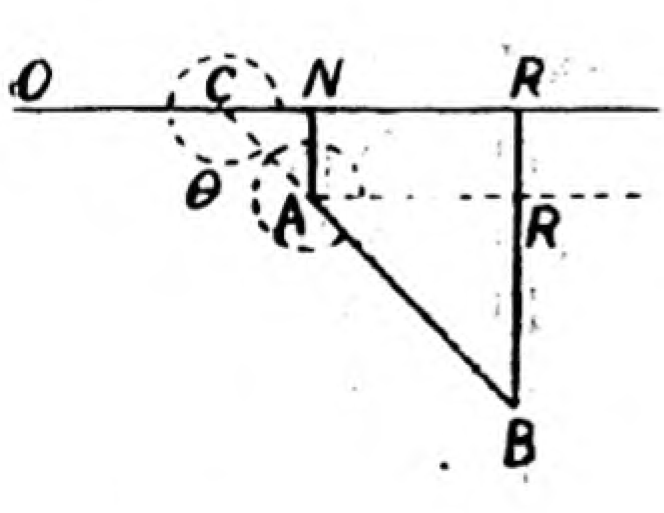
$$\cos 36°=\frac{AF}{AE}=\frac{c}{4x}=\frac{1}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{4}.$$

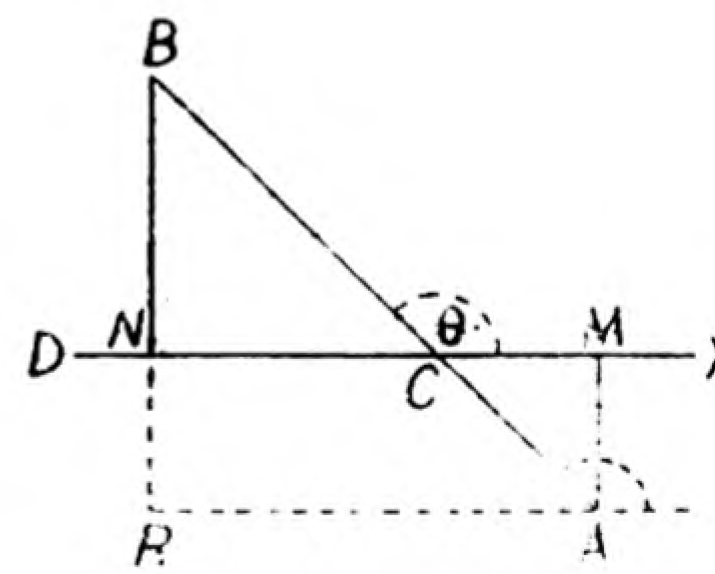
Projection

113. Let AB be any straight line and from its ends A, B, let perpendiculars be drawn to a fixed straight line OX, meeting it in M and N. Then MN is called the *projection* of AB on OX.









If MN be in the same direction as OX, it is positive; if in the opposite direction, it is negative.

114. If θ be the angle between any straight line AB and a fixed line OX, the projection of AB on OX is AB cos θ.

Through A draw a straight line AB parallel to OX and let it meet BN, produced if necessary, in R.

Then in each of the above figures the angle RAB or the angle XCB is equal to θ.

Also MN=AR=AB cos RAB=AB cos θ.

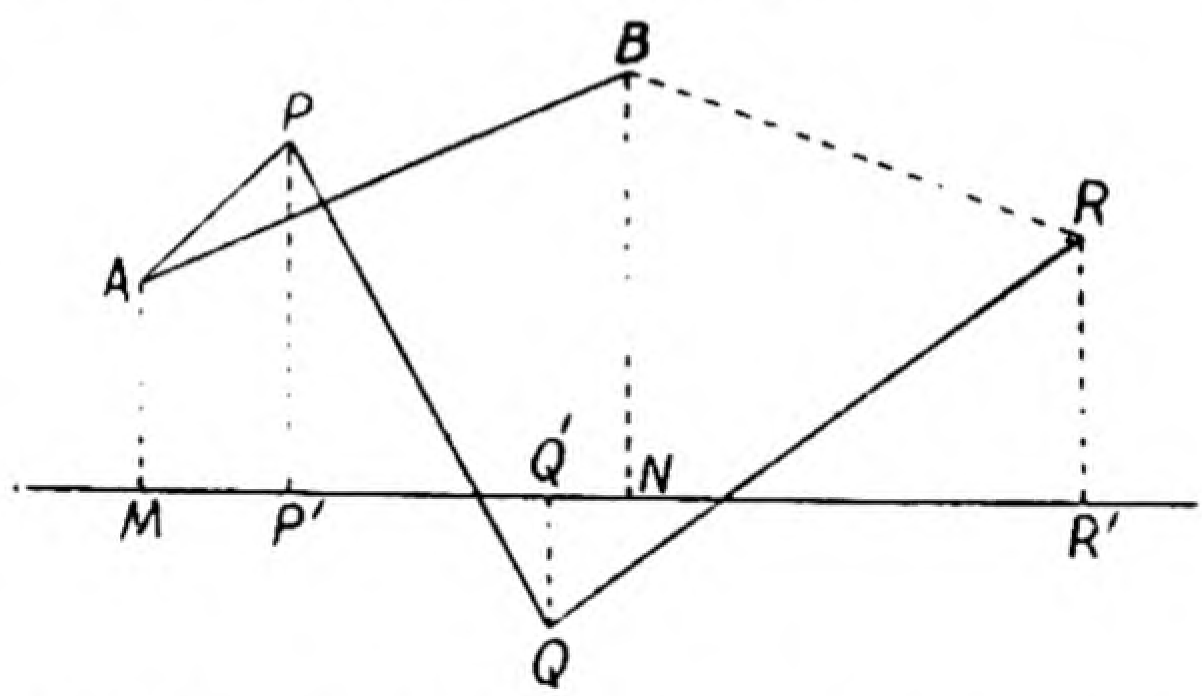
Cor. 1. *It follows that the projection of AB on itself is AB.*

Cor. 2. *It follows that the projection of AB on any line is equal to the projection of an equal line CD parallel to AB in the same sense.*

Similarly it can be shown that the projection of AB on the line OY perpendicular to OX is equal to

$$RB = AB \sin RAB = AB \sin \theta.$$

115. The projection of AB upon any fixed line OX is equal to the sum of the projections on OX of any broken line beginning at A and ending at B.



Let APQRB be any broken line joining AB.
The projection of AP is MP′ and is positive.
The projection of PQ is P′Q′ and is positive.
The projection of QR is Q′R′ and is positive.
The projection of RB is R′N and is negative.

Hence the sum of the projections of the broken line APQRB

$$\begin{aligned} &= MP' + P'Q' + Q'R' + R'N \\ &= MP' + P'Q' + Q'R' - NR' \\ &= MR' - NR' = MN. \end{aligned}$$

Note—A similar proof applies whatever be the positions of A and B and however broken the lines joining them may be.

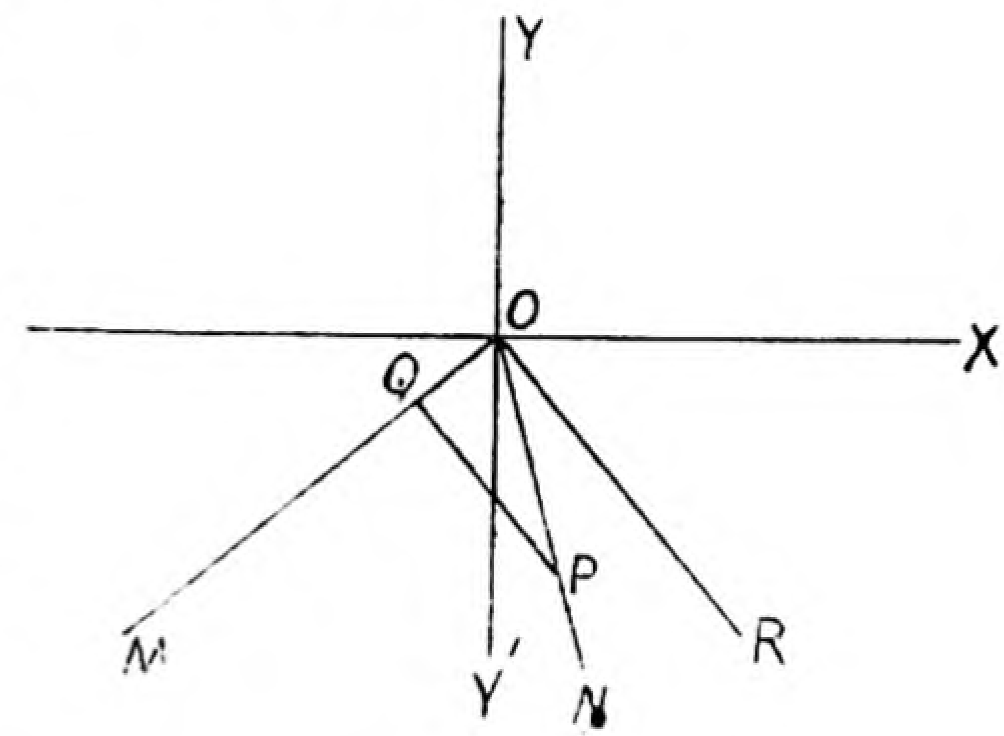
Cor. 1. *The sum of the projections of any broken line joining A to B is equal to the sum of the projections of any other broken line joining the same two points; for, each sum is equal to the projection of the line AB.*

Cor. 2. *Projection of a closed figure on any line is zero.*

116. To prove that

(*i*) $\cos(A+B) = \cos A \cos B - \sin$ si An B.

(*ii*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.



Let the revolving line start from OX and trace out the angle XOM equal to A and then trace out the further angle MON equal to B.

In the final position ON of the revolving line take a point P and draw PQ perpendicular to OM ; also draw OR parallel to QP in the same sense and equal to it.

Projecting on OX, we have

Projection of OP

= projection of OQ + projection of QP

= projection of OQ + projection of OR

$\therefore$ OP cos XOP

$= OQ \cos XOQ + OR \cos XOR$

$= OP \cos B \cos XOQ + OP \sin B \cos XOR.$

$\because$ $OQ = OP \cos B$ and $OR = QP = OP \sin B$;

$\therefore$ cos XOP

$= \cos B \cos XOQ + \sin B \cos XOR,$

or $\cos(A+B)$

$= \cos B \cos A + \sin B \cos(90^\circ + A)$

$= \cos A \cos B - \sin A \sin B.$

(*i*) Projecting on OY, we have

Projection of OP

= projection of OQ + projection of QP

= Projection of OQ + projection of OR

$\therefore$ OP cos YOP

$= OQ \cos YOQ + OR \cos YOR$

$=OP \cos B \cos YOQ + OP \sin B \cos YOR$

or $\cos YOP$

$=\cos B \cos YOQ + \sin B \cos YOR.$

Therefore

$\cos(A+B-90°) = \cos B \cos(A-90°) + \sin B \cos A$

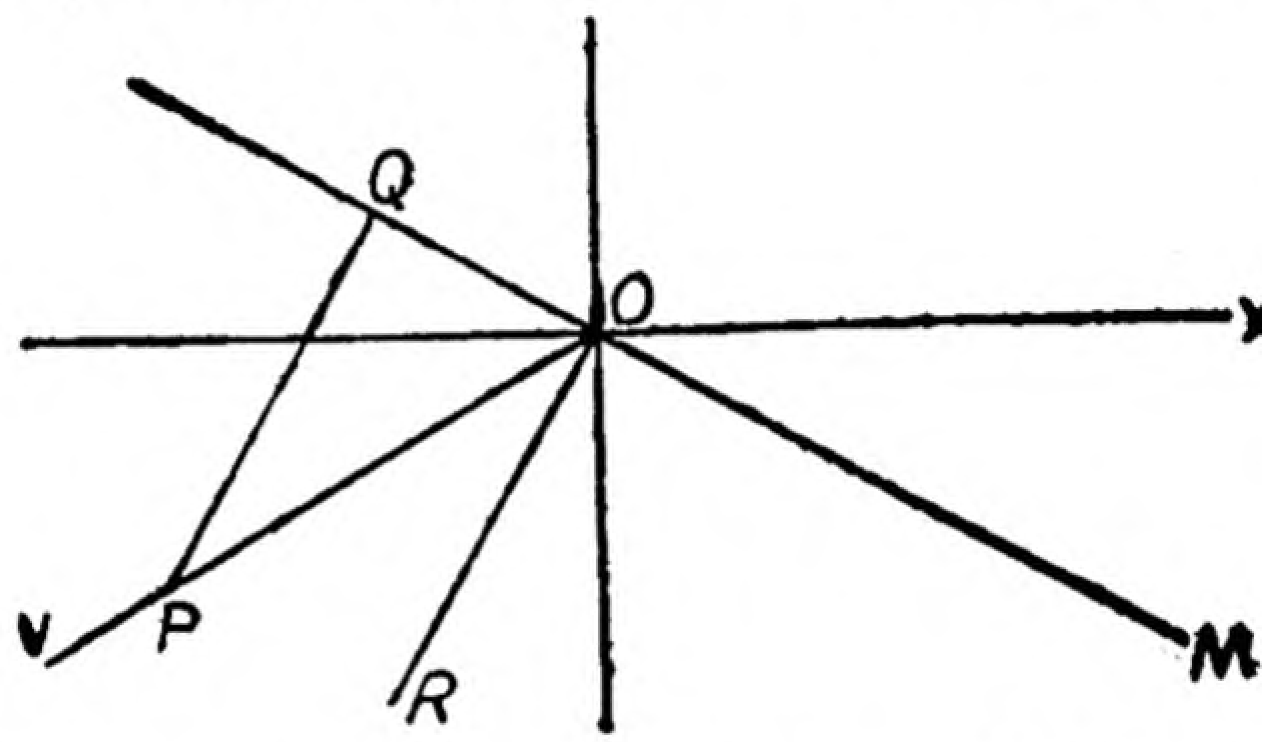
i e, $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Note.—The above method of proof is perfectly general and is applicable to all cases, whatever the angles A, B, and A+B may be.

117. To show that

(*i*) $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

(*ii*) $\sin(A-B) = \sin A \cos B - \cos A \sin B$.



Let the revolving line start from OX and trace out an angle XOM equal to A and then let it revolve backwards and trace out an angle MON equal to B, so that angle XON is A − B

In the final position ON, take a point P and draw PQ perpendicular to MO produced ; also draw OR parallel to QP in the same sense and equal to it.

Projecting on OX, we have

Projection of OP

=projection of OQ+projection of QP.

=projection of OQ+projection of OR.

Therefore $OP \cos XOP$

$= OQ \cos XOQ + OR \cos XOR$

$= OP \cos(180° - B) \cos XOQ$

$+ OP \sin(180° - B) \cos XOR$

$\because \quad OQ = OP \cos(180° - B)$ and $OR = QP$

$= OP \sin(180° - B).$

$\therefore$ cos XOP
$= -\cos B \cos XOQ + \sin B \cos XOR,$
i e., $\cos (A - B)$
$= -\cos B \cos (180^\circ - A) + \sin B \sin (A - 90^\circ)$
$= -\cos B (-\cos A) + \sin B \sin A$
$= \cos A \cos B + \sin A \sin B.$

(*ii*) Projecting on OY, we have

Projection of OP
=projection of OQ+projection of QP
=projection of OQ+projection of OR

Therefore OP cos YOP
$= OQ \cos YOQ + OR \cos YOR$
$= OP \cos (180^\circ - B) \cos YOQ$
$+ OP \sin (180^\circ - B) \cos YOR.$

$\because$ $OQ = OP \cos (180^\circ - B)$
and $OR = OP \sin (180^\circ - B)$

$\therefore$ cos YOP
$= -\cos B \cos YOQ + \sin B \cos YOR$
i.e, $\cos (A - B - 90^\circ)$
$= -\cos B \cos (A - 270^\circ) + \sin B \cos (A - 180^\circ),$

or $\sin (A - B) = -\cos B (-\sin A) + \sin B (-\cos A)$
$= \sin A \cos B - \cos A \sin B.$

Note.—The above method of proof is perfectly general and is applicable to all cases, whatever the angles A, B and A − B may be. It would be an interesting exercise for the student, to draw different figures and to supply the poof for himself.

MISCELLANEOUS EXERCISE III

1. If A, B and C be in arithmetical progression, show that $\dfrac{\cos C - \cos A}{\sin A - \sin C} = \tan B.$

2. If $\cos (A+B) \sin (C+D) = \cos (A-B) \sin (C-D)$, show that $\tan D = \tan A \tan B \tan C.$

3. If the fraction $\dfrac{a \cos (\theta+\alpha) + b \sin \theta}{a' \sin (\theta+\alpha) + b' \cos \theta}$ does not depend on θ, show that $\dfrac{aa' - bb'}{a'b - ab'} = \sin \alpha.$

4. Solve the equation

$$\sin 5\theta - 3\sin 3\theta + 4\sin\theta = 0.$$

5. If $\sin(A+B)\cos C = \sin(A+C)\cos B$, prove that $B-C$ is a multiple of π or A is an odd multiple of $\frac{\pi}{2}$.

6. The sine of an angle is to its cosine as 8 : 15; find their actual values.

7. Find the acute values of θ fom the equation

$$4\sin^2\theta - 2(1+\sqrt{3})\sin\theta + \sqrt{3} = 0.$$

8. (*a*) The angular elevation of a tower at a place A, due south of it is 30° ; and at a place B, due west of A and at a distance a from it, the elevation is 18°. Find the hei of the tower.

(*b*) The angular elevations α, β of the top of a tower are observed at two points A and B. The point A is on the ground ; and B is between A and the tower, at a height b above A and at a horizontal distance a from A. Prove that the height of the tower is $a\dfrac{\sin\alpha\sin(\beta-\theta)}{\cos\theta\sin(\beta-\alpha)}$, where $\tan\theta = \dfrac{b}{a}$.

9. Solve the equation $\sin 3\theta = \sin\theta\cos\theta$.

10. Solve the equation $\cos 4x + \cos 2x + \cos x = 0$.

11. If $A+B+C+D=360°$, show that

$$\sin A + \sin B + \sin C + \sin D = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}.$$

12. Show that in a quadrilateral ABCD,

$$\cos A + \cos B + \cos C + \cos D = 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}.$$

13. If $\sin B = m\sin(2A+B)$ prove that

$$\tan(A+B) = \frac{1+m}{1-m}\tan A.$$

14. If $A+B+C=(2n+1)\pi$, show that

$$\sin 2A + \sin 2B + \sin 2C = 4\cos A\cos B\cos C.$$

15. If $A+B+C=\frac{\pi}{2}$, show that

$$\sin^2 A+\sin^2 B+\sin^2 C=1-2\sin A\sin B\sin C.$$

16. If $2\cos\theta=\sqrt{1-\sin 2\theta}-\sqrt{1+\sin 2\theta}$, show that θ must lie between $(8n+5)\dfrac{\pi}{4}$ and $(8n+7)\dfrac{\pi}{4}$.

17. In any triangle ABC show that

(*i*) $\cos A\cos B=\dfrac{2(a+b)}{c}\sin\dfrac{C}{2}$.

(*ii*) $4R\cos C=r+r_1+r_2-r_3$.

18. In any triangle ABC, show that

(*i*) $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$.

(*ii*) $\dfrac{1}{r_3}-\dfrac{1}{r_2}=\dfrac{c-b}{rs}$.

19. In the triangle ABC, $a=6{\cdot}1$ ft., $b=4{\cdot}5$ ft., and $B=38°$, find the other angles, having given that

$$\log 61=1{\cdot}78533,\ \log 45=1{\cdot}65321.$$

L sin 56° 34′=9·92144, L sin 56° 45′=99·2152.

L sin 38°=9·78934.

20. A boy is sailing his model boat on a circular pond of 440 ft. circumference. The boat takes a straight course along a chord PQ, 120 ft. long. Find to the nearest foot the length of the arc PQ. (take $\pi=\frac{22}{7}$).

21. The sides of a triangle are 50, 36 and 28; find the greatest angle.

22. If the equation $a\cos\theta+b\sin\theta=c$ is satisfied for $\theta=\theta_1$ and $\theta=\theta_2$, show that

$$\sin(\theta_1+\theta_2)=\frac{2ab}{a^2+b^2}.$$

23. The sides of a triangle are x^2+x+1, $2x+1$, and x^2-1; find the greatest angle.

24. The elevation of a tower at a place A due south of it is $\alpha°$, and at a place B due west of A and at a distance a from it, the elevation is $\beta°$. Find the height of the tower.

25. A tower 51 ft. high has a mark at a height of 25 ft. from the ground, find at what distance the two parts subtend equal angles to an eye at the height of 5 ft. from the ground.

26. If $\sin\theta=\dfrac{m^2-n^2}{m^2+n^2}$ find $\tan\dfrac{\theta}{2}$.

27. If $\tan A+\sin A=m$, $\tan A-\sin A=n$, show that $(m^2-n^2)^2=16mn$.

28. In any triangle ABC, show that

$$\frac{a-b}{a}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}} \text{ and } \frac{a+b}{c}=\frac{1+\tan\frac{A}{2}\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}.$$

29. Solve for θ:
$\cos\theta-\sin\theta=\cos\alpha-\sin\alpha$.

30. Solve for θ:
$\cos^3\theta\sin 3\theta+\sin^3\theta\cos 3\theta=\frac{3}{4}$.

31. In any triangle ABC show that

$$\frac{1}{a}\cos^2\frac{A}{2}+\frac{1}{b}\cos^2\frac{B}{2}+\frac{1}{c}\cos^2\frac{C}{2}=\frac{(a+b+c)^2}{4abc}.$$

32. If $A+B+C=(2n+1)\pi$, prove that
$\sin^2 2A+\sin^2 2B+\sin^2 2C=2-2\cos 2A\cos 2B\cos 2C$.

33. In any triangle

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{b+c-a}\cot\frac{A}{2}.$$

34. In a triangle ABC, $\dfrac{\sin A}{\sin B}=\dfrac{m}{n}$ and $\dfrac{\cos A}{\cos B}=\dfrac{p}{q}$,

show that $\dfrac{\tan(B-A)}{\tan C}=\dfrac{mp-nq}{nq+mp}$.

35. ABC is a triangle; a new triangle is formed by the external bisectors of the angles. Show that the sides of the new triangle are

$$4R\cos\frac{A}{2},\ 4R\cos\frac{B}{2} \text{ and } 4R\cos\frac{C}{2}.$$

36. Walking down a hill inclined to the horizon at an angle θ a man observes an object in the horizontal plane whose angle of depression is α. Half way down the hill the angle of depression is β. Prove that $\cos\theta=2\cot\alpha-\cot\beta$.

37. The angle of elevation of the top of a steeple is 58° from a point in the same level at its base, and is 44° from a

point 42 feet directly above the former point. Find to the nearest foot the height of the steeple, given that $\tan 58^\circ = 1{\cdot}600$ and $\tan 44^\circ = {\cdot}965$.

38. If $\dfrac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)} = \dfrac{\tan\gamma}{\tan\beta}$ then prove that
$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0.$ (B. U.)

[Hint : - Apply Componendo and Dividendo.]

39. If $\tan^2\alpha = 1 + 2\tan^2\beta$, show that
$$\cos 2\beta = 1 + 2\cos 2\alpha.$$

40. If $(1+\cos A)(1+\cos B)(1+\cos C)$.
$$=(1-\cos A)(1-\cos B)(1-\cos C),$$
show that each of them is equal to $\sin A \sin B \sin C$.

41. If $\tan(\beta+\gamma) = l\tan\alpha$, $\tan(\gamma+\alpha) = m\tan\beta$, and $\tan(\alpha+\beta) = n\tan\gamma$ and $(m-n)\tan\alpha + (n-l)\tan\beta + (l-m)\times \tan\gamma = 0$, then show that $\dfrac{m-n}{l} + \dfrac{n-l}{m} + \dfrac{l-m}{n} = 0$.

42. If $\tan\theta\tan\phi = \sqrt{\dfrac{a-b}{a+b}}$ show that $(a-b\cos 2\theta)(a-b\cos 2\phi)$ does not depend on θ and ϕ.

43. Show that $\tan 67^\circ\ 30' = 1 + \sqrt{2}$.

44. If $\tan A + \tan 2A = \tan 3A$ show that A must be a multiple of 60° or 20°.

45. If $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, prove that $\sqrt{2}\sin\theta = \sin\alpha - \cos\alpha$.

46. In a triangle ABC right-angled at C, show that
$$\frac{\sin^2 A}{\sin^2 B} - \frac{\cos^2 A}{\cos^2 B} = \frac{a^4 - b^4}{a^2 b^2}.$$

47. Prove that $\dfrac{1+\tan^2\left(\dfrac{\pi}{4}-\theta\right)}{1-\tan^2\left(\dfrac{\pi}{4}-\theta\right)} = \operatorname{cosec} 2\theta.$

48. If $\cos\theta(1+\sin\theta) = 4m$, and $\cot\theta(1-\sin\theta) = 4n$, show that $(m^2-n^2)^2 = mn$.

49. Prove that $\cos 6^\circ \cos 36^\circ \cos 42^\circ \cos 78^\circ = \dfrac{1}{16}$.

50. In any triangle ABC, prove that

$$\frac{b^2-c^2}{a}\cos A+\frac{c^2-a^2}{b}\cos B+\frac{a^2-b^2}{c}\cos C=0.$$

51. If $a\sin\theta-b\cos\theta=0$, show that

$$a\cos 2\theta-b\sin 2\theta=a.$$

52. If $2\tan B\,(1-n\sin^2 A)=n\sin 2A$, show that an $(A-B)=(1-n)\tan A$.

53. If $xy+yz+zx=1$ prove that

(*i*) $\dfrac{x}{1+x^2}+\dfrac{y}{1+y^2}+\dfrac{z}{1+z^2}=\dfrac{2}{\sqrt{[(1+x^2)(1+y^2)(1+z^2)]}}$.

(*ii*) $4yz(1-x^2)+4xz(1-y^2)+4xy(1-z^2)$
$=(1-x^2)(1-y^2)(1-z^2)+(1+x^2)(1+y^2)(1+z^2)$.

54. Indicate a method for solving the equation

$\tan x=2-\dfrac{4}{\pi}x$ graphically, x being measured in radians.

55. If $x^2\cos\alpha\cos\beta+x(\sin\alpha+\sin\beta)+1=0$
and $x^2\cos\beta\cos\gamma+x(\sin\beta+\sin\gamma)+1=0$,
prove that $x^2\cos\alpha\cos\gamma+x(\sin\gamma+\sin\alpha)+1=0$.

56. If $A+B+C=180°$, show that

$$\tan nA+\tan nB+\tan nC=\tan nA\tan nB\tan nC,$$

where n is any integer. Deduce that if $x+y+z=xyz$, then $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$.

57. If θ be an acute angle, find its value from the equation $3\tan\theta+\cot\theta=5\operatorname{cosec}\theta$.

58. Given that $\tan\theta=\dfrac{1+\sqrt{1+a}}{1+\sqrt{1-a}}$ prove that $\sin 4\theta=a$.

59. Solve the equation

$$\tan\theta+\tan\left(\frac{\pi}{2}+\theta\right)=2.$$

60. Prove that in any triangle ABC,

(*i*) $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$.

(*ii*) $\sin^5 A\sin(B-C)+\sin^5 B\sin(C-A)+\sin^5 C\sin(A-B)+\sin A\sin B\sin C\sin(A-B)\sin(B-C)\sin(C-A)=0$.

61. In any triangle ABC circles are inscribed in the angles so that each touches two sides of the triangles and

the inscribed circle ; if r', r'', r''' be the radii of these circles. prove that

$$r' \cot^2\left(\frac{\pi-A}{4}\right)=r'' \cot^2\left(\frac{\pi-B}{4}\right)=r''' \cot^2\left(\frac{\pi-C}{4}\right)=r.$$

62. If a circle and a regular polygon of n sides be concentric, and the area of the polygon outside the circle is equal to the area of the circle outside the polygon, show that if 2θ be the angle subtended at the centre by the portion of a side of the polygon intercepted by the circle then

$$\cos^2\theta=\frac{\pi}{n} \cot \frac{\pi}{n}.$$

63. If R, r denote the radii of the circumscribed and inscribed circles to a regular polygon of any number of sides ; R$'$, r' the corresponding radii to a regular polygon of the same area and double the number of sides, prove that

$$R'=\sqrt{Rr} \text{ and } r'=\sqrt{\frac{r(R+r)}{2}}.$$

64. Given that $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$, determine the cosine of 9° correct to two decimal places.

65. If H is the orthocentre of a triangle ABC, prove that the circumcircles of triangles, BHC, CHA, AHB and ABC are equal.

66. Prove that in any triangle ABC

$\cos A+\cos B+\cos C=\frac{R+r}{R}$, and hence show that if $R=2r$, the triangle must be equilateral.

67. If $\theta=\frac{\pi}{4}$, show that $\cos 3\theta-\cos^2\theta+\cos\theta=\frac{1}{2}$.

68. Prove that in any triangle ABC,

$$\Delta=Rr(\sin A+\sin B+\sin C).$$

69. Prove that in any triangle ABC,

$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$, where p_1, p_2 and p_3 are the three altitudes of the triangle.

70. Prove that in any triangle ABC right-angled at C

$$\tan \frac{A}{2} = \frac{a-b+c}{a+b+c}.$$

71. ABC is an isosceles triangle having B=C; show that the radius of the inscribed circle is

$$\frac{b \sin \frac{A}{2}}{\sin\left(\frac{\pi}{4}+\frac{A}{2}\right)}.$$

72. In the ambiguous case, a, b and A being given, $(b>a)$ if C_1, C_2 be the two values of the angle C, prove that

$$\cos C_1 + \cos C_2 = \frac{2b}{a} \sin^2 A \text{ and}$$

$$1 + \cos C_1 \cos C_2 = \frac{(a^2+b^2) \sin^2 A}{a^2}.$$

73. In the ambiguous case a, b, A being given $(b>a)$, the two values of the third side are c_1 and c_2 $(c_1>c_2)$, show that

(i) $c_1 c_2 = b^2 - a^2$. (ii) $c_1 + c_2 = 2b \cos A$.

(iii) $c_1 - c_2 = 2\sqrt{a^2 - b^2 \sin^2 A}$.

74. The side AB of a triangle ABC is divided at P, so that

$\frac{AP}{PB} = \frac{m}{n}$. If $\angle CPB = \theta$, prove that

$$(m+n) \cot \theta = n \cot A - m \cot B.$$

Also if $\angle ACP = \alpha$ and $\angle BCP = \beta$, show that

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta.$$

75. In any triangle ABC show that

$$\frac{b}{b+c} = \frac{\sin B}{2 \sin\left(B+\frac{A}{2}\right) \cos \frac{A}{2}}.$$

76. If $m = \tan \theta - \sin \theta$ and $n = \tan \theta + \sin \theta$, prove that $m^4 + n^4 = m^2 n^2 + 16mn$.

77. Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of n sides

78. Show that in any triangle ABC,

$$\sin 2m\,A+\sin 2m\,B+\sin 2m\,C$$
$$=(-1)^{m-1}\,4\sin m\,A\sin m\,B\sin m\,C.$$

79. If AD is the median of the triangle ABC, show that

$$\tan ADB=\frac{2bc\sin A}{b^2-c^2}.$$

80. If AD is a median of the triangle ABC, show that

(*i*) $\cot BAD-\cot B=2\cot A.$

(*ii*) $2\cot ADC=\cot B-\cot C.$

81. If the escribed circle corresponding to A be equal to the circumcircle, show that

$$\cos A=\cos B+\cos C.$$

82. In a triangle ABC if
$(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B)$,
show that the triangle is either isosceles or right angled.

83. Show that $\tan^3\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{1-\sin x}{1+\sin x}\cdot\frac{\cos x}{1+\sin x}$ and hence find all the real solutions of the equation,

$$1+\sin x=\sqrt{\cos x(1-\sin x)}.$$

84. If $A+B+C=\pi$, show that $\sin^3 A+\sin^3 B+\sin^3 C$

$$=3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

85. Prove that the area of a quadrilateral is equal to $\frac{1}{2}$ (product of the diagonals) $\times$ (sine of the included angle).

86. Prove that if ABCD be a cyclic quadrilateral, then

$$a^2+d^2-b^2-c^2=(ad+bc)\cos A$$

and $$4S=2(ad+bc)\sin A.$$

Hence obtain the area of a cyclic quadrilateral in terms of the sides.

87. Show that in the cyclic quadrilateral ABCD,

$$BD^2=\frac{(ab+cd)(ac+bd)}{ad+bc}, \text{ and } AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}.$$

88. Show that if A, B, C, D are concyclic, then the circumradius $=\frac{1}{4S}\sqrt{(ab+cd)(ac+bd)(ad+bc)}$.

89. If a circle can be inscribed in a cyclic quadrilateral, show that the radius of the circle is

$$\frac{2\sqrt{abcd}}{a+b+c+d}.$$

90. The area of a circle of radius a is bisected by an arc of a circle, of radius $2a\cos\theta$, which has its centre on the circumference of the first circle. Prove that if θ is in circular measure $2\theta\cos 2\theta-\sin 2\theta+\frac{\pi}{2}=0$.

91. AB is a chord of a circle of radius R and subtends an angle 2θ at the centre O. Prove that the radius of the circle inscribed in the triangle OAB is $R\tan\theta\,(1-\sin\theta)$.

92. Show that in a triangle ABC

$$d^2=R^2(3-2\cos A-2\cos B-2\cos C),$$

where d is the distance between the incentre and the circumcentre.

93. If $\sin\theta+\sin\phi=a$ and $\cos\theta+\cos\phi=b$, prove that

$$\sin(\theta+\phi)=\frac{2ab}{a^2+b^2}.$$

94. A plane is inclined at an angle α to the horizontal plane. A line is drawn in the inclined plane making an angle θ with the common section of the two planes. If ϕ be the inclination of the line to the horizontal plane, prove that

$$\sin\phi=\sin\alpha\sin\theta.$$

What is the greatest value of ϕ if θ varies?

95. Two straight lines, OA, OB of length a and b respectively are inclined to each other at an angle α and st. lines AP, BP, are drawn at rt. angles to OA and OB; prove that if P falls within the angle α, the area of the quadrilateral is

$$\frac{2ab-(a^2+b^2)\cos\alpha}{2\sin\alpha}.$$

PANJAB UNIVERSITY PAPERS

1944

1. (*a*) What is the (*i*) Sexagesimal System (*ii*) *Centesimal* System (*iii*) Circular System ?

(*b*) The sum of two angles is 80 grades and their difference is 18 degrees. Find the angles in degrees.

(*c*) Express in circular measure and also in degrees the angle of a regular polygon of 40 sides.

2. (*a*) Find general expression for all angles having the same cosine.

(*b*) Solve $3 \tan \theta + \cot\theta = 5 \operatorname{cosec}\theta$.

(*c*) Solve $\cos(2x+3y)=\frac{1}{2}$, $\cos(3x+2y)=\frac{\sqrt{3}}{2}$.

3. (*a*) Prove geometrically

$$\cos(A-B)=\cos A \cos B+\sin A \sin B.$$

(*b*) Prove $\cos\beta \cos(2\alpha-\beta)=\cos^2\alpha-\sin^2(\alpha-\beta)$.

(*c*) Prove

$$\cos 2\theta \cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)=\cos(2\theta+2\phi).$$

4. In any triangle ABC prove that

$$\cos A=\frac{b^2+c^2-a}{2bc}.$$

(*b*) In triangle ABC, b, c, B are given, also $b<c$, show that $(a_1-a_2)^2+(a_1+a_2)^2 \tan^2 B=4l^2$, where a_1, a_2 are two values of third side.

5. (*a*) Prove that $\log_a m^n = n \log_a m$.

(*b*) Given $\log 2 = \cdot3010300$, $\log 4844544 = 6\cdot6852530$, find the value of $(54\frac{1}{10}) \times \sqrt{(\cdot256)^3} \div 30$.

(*c*) The hypotenuse of a rt. angled triangle is $3\cdot141024$ and one side is $2\cdot593167$; find the other side.

6. (*a*) Trace the changes in the values of secant x between -2π and 2π, and draw the graph between these limits.

(*b*) A pole, 100 feet high stands in the centre of an equilateral triangle which is horizontal. From the top of the pole each side subtends an angle of $60°$; prove that the length of the side of triangle is $50\sqrt{6}$ feet.

. In any triangle ABC. where r_1 is the radius of e-circle opposite $\angle A$, prove that :—

(*i*) $r_1 = s \tan\frac{A}{2}$, where $2s = a+b+c$.

(*ii*) $(r_1+r_2)\tan\frac{C}{2} = (r_2-r_1)\cot\frac{C}{2} = c$.

(*iii*) $r_2r_3 + r_3r_1 + r_1r_2 = s^2$, where $2s = a+b+c$.

1945

1. (*a*) Show that for all values of θ, $\sec^2\theta = 1+\tan^2\theta$.

(*b*) If $\tan\theta = \frac{1}{\sqrt{3}}$ find the other trigonometic ratios.

(*c*) rove that identity $\frac{1}{\sec\theta+\tan\theta} = \frac{1-\sin\theta}{\cos\theta}$.

2. Establish the following:—

(1) $\cos(A+B) = \cos A \cos B - \sin A \sin B$,

(2) $\cot(180^\circ \pm \theta) = \pm\cot\theta$.

(3) $\log\frac{x}{y} = \log x - \log y$.

3. (*a*) Determine a general expression for all angle having the same sine.

(*b*) Solve (1) $\cos^2 x + \sin x = 1$.

(2) $\sin 3x + \sin 2x + \sin x = 0$.

4. (*a*) Draw the graph of $\sin x$ as x varies from $-\pi$ to π and locate on the graph the values of x for $\sin^2 x = \frac{1}{4}$.

(*b*) Prove that the area of a circle is π (radius)^2.

5. (*a*) Find the value of sin 18°.

(*b*) If $A+B+C=\pi$, show that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(*c*) Tha angle of elevation of the top of a pole is 15° from a point on the ground. On walking 100 feet towards the pole the angle is found to be 30°. Find the height of the pole.

6. (*a*) In any triangle ABC, prove that

$$(1)\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

$$(2)\ \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

(*b*) Solve the triangle ABC, given $a=723{\cdot}4$, $b=547{\cdot}4$ and $c=59°34'$.

7. (*a*) Show that the inradius of a triangle ABC is given by $\frac{\Delta}{s}$ (where Δ denotes the area and s the semi-perimeter).

And that the circumradius by

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}.$$

(*b*) Given $a=58{\cdot}6$, $b=64{\cdot}3$ and $c=52{\cdot}5$, calculate the area of the inscribed circle.

1946

1. (*a*) Prove that for all values of θ

$$\cos^2\theta+\sin^2\theta=1.$$

If $5\sin^2\theta=1$, find the other trigonometric ratios.

(*b*) Show that Radian is a constant angle; and express its magnitude in sexagesimal measure.

2. Establish the following:—

(1) $\sin(A-B)=\sin A\cos B-\cos A\sin B$.

(2) $\tan 3x=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$.

(3) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, if A, B and C are the angles of a triangle.

3. (*a*) Draw the graph of $\tan x$ as x varies from 0 to 2π and locate on the graph the values of x for

(*i*) $3\tan^2 x=1$.

(*ii*) $\tan x=\cot x$.

(*b*) From the top of a cliff, 300 feet high, the angles of depression of the top and bottom of a tower are observed to be 32° 35′ and 62° 16′. Find the height of the tower.

4. (*a*) (*i*) In any triangle ABC, prove that

$$\sin A/2 = \sqrt{\frac{(s-a)(s-c)}{bc}},$$

where $2s = a+b+c$,

and (*ii*) deduce the value of sin A in terms of the three sides.

(*b*) Given sides $b=15$

$c=25$

and the angle B=32° 15′, solve the triangle.

5. (*a*) Solve the following equations :—

(1) $\cos 3x + \cos 2x + \cos x = 0$.

(2) $11^{4x-5} \times 3^{2x} = 5^{3-x} \div 7^{-x}$

(*b*) Using $\text{Lt} \dfrac{\sin\theta}{\theta} = 1$,

for very small angles, determine the approximate value of cos 7′30″ to four places of decimals.

6. (*a*) Prove that the radius of the escribed circle opposite to the angle A of the △ABC is given by

$$r_1 = \frac{\Delta}{s-a} = s \tan A/2,$$

where △ denotes the area of the △.

(*b*) If the sides of a triangle are

$a=60{\cdot}1$

$b=65{\cdot}4$

$c=52{\cdot}7$,

calculate the area of the escribed circle.

SPECIMEN SOLVED PAPER :

EMERGENCY EXAMINATION 1948

1. (*a*) Define $\sin\theta$ for all values of θ, and prove that

$$\sin^2\theta+\cos^2\theta=1.$$

If $3\cos^2\theta=1$, find the other trigonometric ratios.

(*b*) Prove that

$$(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1.$$

Sol. (*a*) Book Article.

$$3\cos^2\theta=1$$

$$\cos^2\theta=\tfrac{1}{3}$$

$$\cos\theta=\pm\frac{1}{\sqrt3}$$

$$\sin\theta=\frac{\sqrt2}{\sqrt3},\ \cot\theta=\frac{1}{\sqrt2},\ \sec\theta=\sqrt3$$

$$\tan\theta=\sqrt2,\ \operatorname{cosec}\ \theta=\frac{\sqrt3}{\sqrt2},\qquad \cos\theta=\frac{1}{\sqrt3}.$$

(*b*) $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$

$$=\left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$=\frac{1-\sin^2\theta}{\sin\theta}\cdot\frac{1-\cos^2\theta}{\cos\theta}\cdot\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{\cos^2\theta\,.\,\sin^2\theta}{\sin\theta\,.\,\cos\theta}=\frac{1}{\sin\theta\cos\theta}=1.$$

2. (*a*) Determine the general expression for all angles having the same cosine.

(*b*) Solve the equations :

(*i*) $2\sin^2 x+\sqrt3\cos x+1=0.$

(*ii*) $\tan n\theta=\cot m\theta.$

Sol. (*a*) Book Article

(*b*) (*i*) $2\sin^2x+\sqrt3\cos x+1=0$

$$2-2\cos^2x+\sqrt3\cos x+1=0$$

$$2\cos^2 x-\sqrt3\cos x-3=0$$

$$\cos x = \frac{\sqrt{3} \pm \sqrt{3+24}}{4} = \frac{\sqrt{3} \pm 3\sqrt{3}}{4}$$

$$\therefore \cos x = \sqrt{3} \text{ or } -\frac{\sqrt{3}}{2}.$$

Rejecting the value $\sqrt{3}$ we get $\cos x = -\frac{\sqrt{3}}{2}$

$$\therefore \quad x = 15^\circ = \frac{5\pi}{6}.$$

Hence the general value of

$$x = 2n\pi \pm \frac{5\pi}{6}.$$

(*ii*) $\tan n\theta = \cot m\theta$

$$= \tan\left(\frac{\pi}{2} - m\theta\right)$$

$$\therefore \quad n\theta = k\pi + \frac{\pi}{2} - m\theta$$

$$\theta(m+n) = k\pi + \frac{\pi}{2}$$

$$\therefore \quad \theta = \frac{(2k+1)\pi}{2(m+n)}.$$

3. Prove that

(*i*) $\sin(A-B) = \sin A \cos B - \cos A \sin B.$

(*ii*) $\sin(90+\theta) = \cos\theta.$

(*iii*) $\cos 20, \cos 40, \cos 60, \cos 80 = \frac{1}{16}.$

Sol III (*i*) Book Article.

(*ii*) Book Article.

(*iii*) Solved example page 113.

4. (*a*) In any triangle ABC, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

(*b*) Given $b=16$, $c=25$, and $B=33^\circ 15'$, solve the triangle.

Sol. (*a*) Book Article.

(*b*) $\angle B = 33° \, 15'$

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\sin C = \frac{c \sin B}{b}$$

$$L \sin C = \log c - \log b + L \sin B$$
$$= \log 25 - \log 16 + L \sin 33° \, 15'$$
$$= 1{\cdot}3979 - 1{\cdot}2041 + 9{\cdot}7390$$
$$= 9{\cdot}9328 = L \sin 58° \, 57'$$

$\therefore \quad C = 58° \, 57'$

$\therefore \quad \angle C_1 = 58° \, 57'$ and $C_2 = (180 \; 58°, \; 57' = 122° \, 3'$.

$$\angle A_1 = 180° - (B + C_1) = 180° - (92° \, 12')$$
$$= 87° \, 48'$$

$$\angle A_2 = 180° - (B + C_2) = 180° - (33° \, 15' + 122° \, 3')$$
$$= 24° \, 44'$$

To find side a_1

$$\frac{a_1}{\sin A_1} = \frac{b}{\sin B}$$

$$\log a_1 = \log b + L \sin A - L \sin B$$
$$= \log 16 + L \sin 87° \, 48' - L \sin 33° \, 15'$$
$$= 1{\cdot}2041 + 9{\cdot}9947 - 9{\cdot}7390$$
$$= 1{\cdot}4648$$
$$= \log 29{\cdot}16$$

$\therefore a_1 = 29{\cdot}16.$

5. (*a*) Prove that if θ is measured in radians

$$\underset{\theta \to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta} = 1.$$

(*b*) Find the area of a regular polygon of n sides inscribed in a circle and radius r, and deduce the area of the circle.

Sol. (*a*) Book Article (*b*) Book Article.

6. (*a*) Trace the variations in the value of $\tan\theta$ as θ changes from 0° to 360° and draw its graph.

(*b*) If $A+B+C=\pi$, Show that

(*i*) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(*ii*) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

Sol.

(*a*) Book Article

(*i*) $\sin A+\sin B+\sin C$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\sin\frac{C}{2}\right\}\quad\because\sin\frac{A+B}{2}=\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}\quad\because\sin\frac{C}{2}=\cos\frac{A+B}{2}$$

$$=2\cos\frac{C}{2}\,2\cos\frac{A}{2}\cos\frac{B}{2}$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(*ii*) $\cos A+\cos B+\cos C=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$$+1-2\sin^2\frac{C}{2}$$

$$=1+2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\sin\frac{C}{2}\right\}\quad\because\cos\frac{A+B}{2}=\sin\frac{C}{2}$$

$$=1+2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}$$

$$=1+2\sin\frac{C}{2}\left\{2\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

ANSWERS

EXERCISE I, Page 13.

$(\pi = \frac{22}{7})$.

1. (*i*) 2nd. (*ii*) 2nd. (*iii*) 3rd. (*iv*) 1st.
2. (*i*) $72^g\ 91'\ 66{\cdot}6''$ (*ii*) $83^g\ 33'\ 33\frac{1}{3}''$.
3. (*i*) $\frac{\pi}{12}$ (*ii*) $\frac{3\pi}{40}$ (*iii*) $\frac{7\pi}{12}$ (*iv*) $\frac{27\pi}{40}$.
4. $\frac{4\pi}{15}$, $\frac{\pi}{3}$, $\frac{2\pi}{5}$. 5. $\frac{2\pi}{9}$. $\frac{\pi}{3}$, $\frac{4\pi}{9}$; 40°, 60°, 80°.
6. $\frac{\pi}{5}$. 7. 40°, 50°, 90° ; $44^g\frac{4}{9}$, $55^g\frac{5}{9}$, 100^g.
8. 40°, 60°, 80° ; $\frac{2\pi}{9}$, $\frac{\pi}{3}$, $\frac{4\pi}{9}$; $44\frac{4}{9}$, $66\frac{2}{3}$, $88\frac{8}{9}$ grades.
9. 30°, 60°, 90°. 10. $\frac{\pi}{3}$. 11. $\frac{\pi(n-2)}{n}$, $\frac{(n-2)}{n}\,180°$.
12. 6°.

EXERCISE II, Page 15.

1. $2\frac{65}{72}$ feet nearly. 2. ·05236 inch nearly.
3. $\pi - 2$. 4. 5 : 4. 5. 2062·65 ft. nearly.
6. 1·5359 ft. nearly. 7. $\frac{\pi}{8}$. 8. 1 : 400.
9. 9 ft. $6\frac{6}{11}$ inches.

REVISION QUESTIONS I, Page 17.

1. $\frac{5\pi}{6}$. 2. 0·01745, ·00029. 3. 55°, 35°.

. $\frac{5\pi}{36}$, $\frac{\pi}{3}$, $\frac{19\pi}{36}$. 5. 135°. 6. 63. 8. $\frac{13751}{21600}\pi$.

9. 8. 10. 2·2 ft. 11. 170·7 yds. approx,

12. $\frac{4\pi}{35}$, $\frac{9\pi}{35}$, $\frac{14\pi}{35}$. $\frac{19\pi}{35}$, $\frac{24\pi}{35}$. 14. 3·14159 ft.

15. $\frac{1}{3}$ of a radian.

EXERCISE III, Page 23.

6. $\tan^2\theta$. 7. $\sin\theta$. 8. $2\sec^2 A$. 9. 0.
40. Yes ; no.

EXERCISE IV, Page 26.

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 2. $\frac{x}{a}+\frac{y}{b}=1$.

3. $p^2+q^2=\frac{1}{2}$. 4. $\left(\frac{x}{a}\right)^{\frac{2}{n}}+\left(\frac{y}{b}\right)^{\frac{2}{n}}=1$.

5. $(x^2-y^2)^2=16xy$. 6. $y^3=1+x^3$.
7. $1+(3-x)^2=(y-4)^2$.

EXERCISE V, Page 30.

1. (*i*) − (*ii*) + (*iii*) +.

2 $\cot\theta=\frac{1}{x}$, $\sin\theta=-\frac{x}{\sqrt{1+x^2}}$, $\cos\theta=-\frac{1}{\sqrt{1+x^2}}$, $\operatorname{cosec}\theta=-\frac{\sqrt{1+x^2}}{x}$, $\sec\theta=-\sqrt{1+x^2}$, where $\tan\theta=x$.

3. (*i*) Yes (*ii*) No (*iii*) Yes (*iv*) No (*v*) No (*vi*) Yes.

4. Second. 5. Fourth. 6. Third. 7. $\frac{24}{25}$. $\frac{5}{13}$. 9. $-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$. 10. $\frac{10}{\sqrt{3}}$.

11 $\pm\frac{1}{\sqrt{3}}$, according as θ lies in the third or the fourth quadrant.

12. $-\frac{5}{13}, -\frac{5}{12}$. 14. (*i*) $\frac{1}{\sqrt{3}}$ (*ii*) $-\frac{1}{\sqrt{3}}$.

15. $\pm\frac{m^2-1}{2m}, \pm\frac{1+m^2}{m^2-1}$. 16. $\pm\frac{2mn}{m^2+n^2}, \pm\frac{m^2-n^2}{m^2+n^2}$.

17. $\frac{5}{4}, \frac{13}{5}, -\frac{5}{13}$. 18. 2. 19. No. 23. $\frac{1}{3}$, 3.

REVISION QUESTIONS II, Page 33.

1. $\frac{1}{\sin^5\theta}$. 2. $1-\cos^2\theta+\cos\theta$.

4. $\cot\alpha = \frac{7}{24}$, $\sin\alpha = -\frac{24}{25}$, $\operatorname{cosec}\alpha = -\frac{25}{24}$,

$\cos\alpha = -\frac{7}{25}$, $\sec\alpha = -\frac{25}{7}$.

5. $\sec x = \pm\sqrt{2}(3-1)$, $\cos x = \pm\frac{\sqrt{3}+1}{2\sqrt{2}}$,

$\sin x = \pm\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\operatorname{cosec} x = \pm\frac{2\sqrt{2}}{\sqrt{3}-1}$, $\cot x = 2+\sqrt{3}$.

17. 1, $-\frac{1}{5}$.

EXERCISE VI, Page 42.

11. Yes. 12. 3. 13. $0, \pi, 2\pi, \frac{\pi}{3}$. 14. $0, \pi, 2\pi$.

15. $0, \pi, 2\pi, \frac{\pi}{4}$. 16. $\frac{\pi}{4}$. 17. A=45°, B=15°.

18. A=60° ; B=30°. 19. $\frac{\sqrt{6}}{8}$.

EXERCISE VII, Page 45.

1. $\frac{80}{\sqrt{3}}$ ft. 2. $200\sqrt{3}$ ft., 600 ft. 3. $175\sqrt{3}$ yds.

4. $125\sqrt{3}$ ft. 5. 60 ft. 6. 120 ft. 7. $\frac{25}{2}(\sqrt{3}+1)$ ft.

8. $\frac{100}{3}(3-\sqrt{3})$ ft., 100 ft. 9. $\frac{100(3-\sqrt{3})}{3}$ ft.

10. $\frac{5(\sqrt{3}+1)}{2+\sqrt{2}}$ miles, $\frac{5}{2+\sqrt{2}}$ miles.

11. 4·33 miles, 300 miles per hour.

12. 16 ft. 9 in (to the nearest inch.)

EXERCISE VIII, Page 48.

1. 12°,13′. 2. 13°, 14′. 3. 21°, 11′.

4. b=8·29, a=5·592. 5 a=7·1405, c=8·7169 approximately. A=55°.

6. a=2467·0155, c=25057·6037 nearly, B=10°.

7. b=417·8, c=650·026, B=39° 43′.

8. b=186·60 and c=193·18, A=15°.

9. 71° 47′. 10. 12·3755″.

REVISION QUESTIONS III, Page 49.

4. $\frac{\pi}{6}, \frac{\pi}{2}$. 5 1521·23 ft 6. 800 ft.
7. $75\sqrt{3}$ ft. 8. $100(3+\sqrt{3})$ ft.
9. $25(3-\sqrt{3})$ ft. per minute. 10. $10(\sqrt{3}+1)$ ft.
12. $40\sqrt{3}$, 40 ft, 13. 9·334″, 15·1508″.
14. 92·37 sq. inches.

EXERCISE IX, Page 60.

1. (*i*) $-\frac{1}{\sqrt{2}}$. (*ii*) $\frac{1}{\sqrt{2}}$. (*iii*) $-\sqrt{3}$.

2. (*i*) $\frac{1}{2}$. (*ii*) $-\frac{1}{\sqrt{2}}$. (*iii*) $\sqrt{3}$.

3. (*i*) $-\frac{2}{\sqrt{3}}$. (*ii*) -1. (*iii*) $\pm\infty$. 4. $-\sin^2 A$.

5. tan A. 8. 1. 9. 1.

EXERCISE X, Page 83.

4. (*i*) About 17°. (*ii*) About 37°. Read $\sin x = \cdot 6$.
5. (*i*) About $\pm 37°$. (*ii*) $\pm 124°$. 6. 45°, 225°.
7. (*i*) About 26°. (*ii*) About 108°.
11. 72°. 12. $24\frac{1}{2}°$, 114°. 13. 42°, 138°. 14. About 17°.

MISCELLANEOUS EXERCISES I, Page 85.

1. 6. 2. 5°, 37′, 30″ ; 6^g, 25`. 4. 33 ft.

$\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}$. 6. 20°, 60°, 100°. 11. $\sqrt{24}$.

12 $\frac{\pi}{2n}$, $-\frac{1}{\sqrt{2}}$, 1, $-\sqrt{2}$, $\frac{1}{\sqrt{3}}$, 13. $\frac{1}{2}$, $\frac{1}{2}$, 1, -1.

14. $-\frac{1}{2}$, $-\sqrt{3}-\sqrt{2}$, 2. 16. $\frac{1-t^2}{1+t^2}$. 17. t^6.

25. $\frac{\pi}{2n}$.

27, (*i*) 135°, 315°. (*ii*) 120°, 240°. (*iii*) 30°, 150°, 270°.
(*iv*) 30°, 150°. (*v*) 30°, 150°. 210°, 330°.
30. $-\cos 6°$, $-\cos 3°$, $-\tan 19°$, $-\sec 42°$ · $-\cdot 9945$, $-\cdot 9986$, $-\cdot 3443$, $-1\cdot 34$.

31. 1·7 radians. 32. $\pm\frac{a^2-b^2}{2ab}$.

33. 600, $300\sqrt{3}$, $200\sqrt{3}$, $100\sqrt{3}$ ft. 36. $\pm\frac{17}{13}$

EXERCISE XI, Page 92.

8. $\frac{63}{65}$. 10. $\frac{2mn}{m^2+n^2}$, where $m=ad+bc$ and $n=ac-bd$.

12. $-\frac{33}{65}$. 13. $\frac{\sqrt{3}}{2}$. 14. $\frac{56}{33}$. 15. $\sin 5x$.

16. $\cos 8x$. 17. $\cos A$.

EXERCISE XII, Page 97

1. $-\frac{7}{25}$. 2. $\frac{47}{49}$. 3. $\pm\frac{120}{169}$. 4. $-\frac{5}{12}$.

5. $\frac{4}{5}$, $-\frac{3}{5}$. 6. $\frac{1}{\sqrt{2}}$. 7. $\frac{\sqrt{3}}{2}$.

EXERCISE XIII, Page 102.

2. $\frac{24}{25}$, $-7/25$. 8. $\cos A$. 9. $\sin y$.

12. $W(\sin\alpha+\mu\cos\alpha)$. 13. $\frac{2v^2\sin\theta}{g\cos^2 A}\cos(A+\theta)$.

EXERCISE XIV, Page 105.

6. $\frac{s_1-s_3}{1-s_2}$, $\frac{s_1-s_3}{1-s_2-s_4}$.

EXERCISE XV, Page 109.

22. 2. 23. $2\cos\left(\theta-\frac{\pi}{6}\right)$.

24. $\sqrt{a^2+b^2}$. 35. $a=\sqrt{3}$, $b=1$.

EXERCISE XVI, Page 115.

1. $2\sin 3\theta\cos\theta$. 2. $2\cos 4\theta\sin\theta$,
3. $2\cos\frac{\pi}{4}\cos\left(\frac{\pi}{4}-3\theta\right)$ or $2\sin\frac{\pi}{4}\cos\left(\frac{\pi}{4}-3\theta\right)$.
4. $2\sin 6\theta\sin\theta$. 5. $-2\sin 35^\circ\sin 15^\circ$.
6. $2\sin(45^\circ+A)\cos(45^\circ+B)$. 7. $\sin 5\theta-\sin\theta$.
8. $\cos 9\theta+\cos$. 9. $\cos 2\theta-\cos 4\theta$.

REVISION QUESTIONS, Page 118.

11. a/b.

EXERCISE XVII, Page 121.

1. $\cos n\alpha \sin n\alpha \operatorname{cosec} \alpha$.
2. $\sin (n+1)\alpha \sin n\alpha \operatorname{cosec} \alpha$.
3. $\frac{1}{2} \sin (n+1)\alpha \sin n\alpha \operatorname{cosec} \alpha$.
4. $\frac{1}{2} \sin (n+2)\alpha \sin n\alpha \operatorname{cosec} \alpha - \frac{n}{2} \sin \alpha$.
5. $\frac{n}{2} \cos \alpha - \frac{1}{2} \cos (n+2)\alpha \sin n\alpha \operatorname{cosec} \alpha$.

EXERCISE XVIII, Page 123.

1. (*i*) + (*ii*) + (*iii*) −.
2. (*i*) + (*ii*) − (*iii*) − (*iv*) +.
3. (*i*) + (*ii*) − (*iii*) +. 4. $\frac{1}{2}\sqrt{2-\sqrt{2}}, -\frac{1}{2}\sqrt{2+\sqrt{2}}$.
5. $-\frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2}$. 6. $\frac{3}{7}, \frac{3}{\sqrt{58}}, \frac{7}{\sqrt{58}}$.
7. (*i*) +. (*ii*) −. (*iii*) −. 8. (*i*) +. (*ii*) −. (*iii*) −.
9. $\frac{\sqrt{3}-1}{2\sqrt{2}}. \frac{\sqrt{3}+1}{2\sqrt{2}}$. 13. $\frac{a}{b}, \frac{b}{a}$.

MISCELLANEOUS EXERCISES; II. Page 135.

1. $\frac{63}{16}$. 10. $\frac{m+n}{m-n}$ or $\frac{m-n}{m+n}$.
13. $\frac{2ab}{a^2+b^2}$. 19. $\frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}$.
20. $\frac{\sqrt{5}+1}{4}$. 23. $-\frac{1}{\sqrt{2}}$.
25. $16 \cos^5 A - 20 \cos^3 A + 5 \cos A, \frac{\sqrt{10-2\sqrt{5}}}{4}$.
32. $\frac{1}{2}\sqrt{2-\sqrt{2}}$. 33. $a(2c^2-d^2)=bdc$. 60. $e \cos \phi$.

EXERCISE XX, Page 143.

1. (*i*) 60° (*ii*) 45° (*iii*) 30° (*iv*) 54° (*v*) 54°, 8′ (*vi*) 15°, 50′

2. (*i*) 150°, 210°, (*ii*) 60°, 240° (*iii*) 210°, 330°.

3. (*i*) $\frac{\pi}{2}, -\frac{\pi}{2}$. (*ii*) 45°, 135°, (*iii*) 150°, (*iv*) $\frac{}{14}$.

EXERCISE XXII, Page 151.

1. $n\pi+(-1)^n\frac{\pi}{6}$. 2. $n\pi-(-1)^n\frac{\pi}{3}$. 3. $2n\pi\pm\frac{\pi}{4}$.

4. $(2n+1)\pi\pm\frac{\pi}{3}$. 5. $n\pi+\frac{\pi}{4}$.

6. $n\pi+\frac{2\pi}{3}$. 7. $n\pi+\frac{3\pi}{4}$.

8. $n\frac{\pi}{2}+(-1)^n\frac{\pi}{4}$. 9. $2n\frac{\pi}{3}\pm\frac{\pi}{9}$. 10. $\frac{n\pi}{5}-\frac{\pi}{30}$.

11. $n\pi\pm\frac{\pi}{3}$. 12. $n\pi\pm\frac{\pi}{6}$. 13. $n\pi\pm\frac{\pi}{3}$.

15. $2n\pi+\frac{7\pi}{6}$. 16. $2n\pi-\frac{\pi}{6}$.

17. $\left(n+\frac{m}{2}\right)\pi\pm\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$,

$\left(\frac{m}{2}-n\right)\pi\pm\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$.

18. $\left(n+\frac{m}{2}\right)\pi+\frac{\pi}{8}\pm\frac{\pi}{12}$, $\left(n-\frac{m}{2}\right)\pi-\frac{\pi}{8}\pm\frac{\pi}{12}$.

19. $A=(m+n)\frac{\pi}{2}+\frac{5\pi}{22}$, $B=(l-m)\frac{\pi}{2}+\frac{\pi}{24}$ and

$C=(l-n)\frac{\pi}{2}+\frac{\pi}{12}$, where l, m and n are integers.

20. $2n\pi$. 21. $n\pi\pm\frac{\pi}{3}$, $n\pi\pm\frac{\pi}{4}$.

22. $2n\pi\pm\frac{\pi}{3}$, $(2k+1)\pi$. 23. $n\pi+(-1)^n\frac{\pi}{6}$.

24. $n\pi+\frac{\pi}{3}$ or $n\pi-\frac{\pi}{3}$. 25. $n\pi$ or $m\pi+(-1)^m\frac{\pi}{2}$.

EXERCISE XXIII, Page 156.

1. $2n\pi\pm\frac{\pi}{4}$. 2. $2n\pi+\frac{3\pi}{4}$, 3. $n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$.

4. $n\pi+(-1)^n\frac{\pi}{4}+\frac{\pi}{6}$. 5. $n\pi+(-1)^n\frac{\pi}{2}-\frac{\pi}{6}$.

6. $2n\frac{\pi}{3}+\frac{\pi}{12}\pm\frac{\pi}{9}$. 7. $n\frac{\pi}{2}+(-1)^n\frac{\pi}{8}$. 8. $\frac{n\pi}{2-(-1)^n3}$.

9. $\frac{k\pi}{m-(-1)^kn}$. 10. $\frac{(2k-1)\pi}{m\pm n}$. 11. $\frac{k\pi}{m-n}$.

12. $\frac{(2k+1)\pi}{2(m+n)}$. 13. $\frac{n\pi}{3}$ or $n\pi\pm\frac{\pi}{3}$. 14. $\frac{n\pi}{4}\pm\frac{\pi}{24}$, $\frac{n\pi}{8}$.

15. $n\pi$ or $\theta=n\pi+(-1)^n\phi$ where $\sin\phi=\cdot 32$.

16. $n\pi$, $n\pi-\phi$ where $\tan\phi=\frac{1}{2}$. 17. $\frac{n\pi}{2}\pm\frac{\pi}{4}$.

18. $\theta=k\frac{\pi}{2}+(-1)^k\frac{\phi}{2}$, where $\sin\phi=\frac{4}{2n+1}$.

19. $\theta=n\pi\pm\alpha$. 20. $\frac{n\pi}{2}$ or $(2m+1)\pi\pm\frac{\pi}{3}$.

REVISION QUESTIONS VI, Page 157.

1. $60°$. 2. $\frac{\pi}{12}$. 4. $30°$, $150°$, $390°$ and $510°$.

5. $n\pi-\frac{\pi}{4}$ 6. $n\pi\pm\frac{\pi}{6}$.

7. $\theta=n\pi+\phi$ where $\tan\phi=2\pm\sqrt{3}$.

8. $n\pi$ or $n\pi-(-1)^n\frac{\pi}{6}$. 9. $2n\pi+\frac{\pi}{4}$

10. $\frac{n\pi}{4}+(-1)^n\frac{\pi}{12}$, $15°$, $30°$, $105°$, $120°$.

12. $\frac{n\pi+(-1)^n\frac{\pi}{2}}{1+(-1)^n2}$ or $\frac{\frac{\pi}{2}-2n\pi}{\pm 2+(-1)^n}$

13. $\frac{n\pi}{3}$ or $\frac{1}{4}\left(2n\pi \pm \frac{\pi}{3}\right)$.

14. $\theta = 2n\pi \pm \frac{\pi}{2}$, $m\ \frac{\pi}{2} \pm \frac{\pi}{4}$, $p\frac{\pi}{4} \pm \frac{\pi}{8}$.

15. $\frac{n\pi}{8}$ or $\frac{n\pi}{4} \pm \frac{\pi}{24}$. 16. 45°. 18. $-1, \frac{1}{2}$.

REVISION QUESTIONS VII, Page 173.

1. 120°. 5. 60°, 120°.

EXERCISE XXVII, Page 177.

1. $\log_3 243 = 5$. 2. $\log_2 16 = 4$. 3. $\log_{10} \cdot 01 = -2$.
4. $\log_{16} 64 = \frac{3}{2}$. 5. 2. 6. $\frac{1}{2}$.
7. 8. 8. 5.

EXERCISE XXVIII, Page 182.

1. (*i*) 4. (*ii*) $\frac{4}{3}$. (*iii*) $\frac{5}{4}$. 2. $\bar{4}\cdot69897$. (*ii*) $\bar{3}\cdot58146$.
3. 1·1461. 4. 1·6902. 5. 1·9912. 6. ·3890.
7. 2·4771. 8. (*i*) 21. (*ii*) 13. (*iii*) 30.
9. (*i*) 7th. (*ii*) 21st. (*iii*) 32nd. 10. 1·09.
11. ·68. 12. ·6735.

EXERCISE XXIX, Page 188.

1. ·01691. 2. ·2008. 3. ·0006811. 4. 9·29.
5. ·2560. 6. 4·616. 7. 1·4515450. 8. ·401.
9. 3·1900. 10. ·6309. 11. 1·292, ·5.

EXERCISE XXX, Page 191.

1. $x = \cdot667$. 2. A=31° 8′. 3. $b = 26\cdot64$ and $c = 31\cdot57$.
4. A=33° 22′, $a = 16\cdot44$, 24·98. 5. 29·17.
6. 49° 37′, 130° 24′. 7. 22° 29′.

REVISION QUESTIONS VIII, Page 192.

1. 1·585. 2. ·6441. 4. 12, 12th, 197·7.
5. (*i*) 1·536, (*ii*) 444·8.
6. (*i*) ·2838. (*ii*) $\bar{1}\cdot9346$. (*iii*) ·1957. (*iv*) $\bar{1}\cdot9937$.
(*v*) 2·9736. (*vi*) $\bar{1}\cdot2150$. 10. 2·6. 11. $\frac{13 \pm \sqrt{29}}{2}$.
12. 10·02. 13. ·7907.

EXERCISE XXXI, Page 196.

1. 71° 30′. 2. 75° 31′. 3. 132° 35′. 4. 57° 52′.
5. A=33° 40′ ; B=101° 48′ ; C=44° 32′.
6. A=95° 28′, B=56° 52′, C=27° 40′.
7. A=38° 13′, B=60°, C=81° 47′.
8. A=53° 8′, B=59° 30′, C=67° 22′.
9. A=90° 50′, B=41° 24′, C=47° 46′.
10. A=114° 2′, B=36° 46′, C=29° 12′.
11. A=70° 36′ 40″, B=52° 16′, C=57° 8′.
12. A=30° 5′, B=131° 16′, C=17° 54′.
13. 132° 34′.

EXERCISE XXXII, Page 198.

1. B=92° 41′, C=54° 49′, a=5·917.
2. A=66° 38′, C=87° 8′. b=14·35.
3. B=76° 18′, C=41° 25′, a=48·21.
4. A=60° 41′, B=39° 19′ and c=98·35
5. B=129° 29′, C=13° 31′ and a=64·65.
6. B=49° 49′, C=70° 31′
7. A=24° 15′, B=34° 7′ and c=36·48.
8. A=109° 40′, C=19° 88′ and b=559·6.
9. B=64° 23′ ; C=72° 43′ ; a=18·92.
10. B=118° 37′. C=31° 45′ ; a=20·95.
11. b=61·83 or 165·8 ; A=54° 21′ or 17° 29′ ; B=39° 32′ or 140° 30′.
12. 51° 12′, 26°. 13. 1° 50′.

EXERCISE XXXIII, Page 200.

1. A=66° 40′, b=237, c=1·581. 2. 20·98
3. 403·5, 4. a=21·42, b=22·34 and C=24° 24′.
5. A=42° 54′ ; b=25·07, c=26·56.
6. b=37·3, c=22·3, A=29° 38′.
7. C=87° 8′ ; a=298· b=14·35
8. 62 ft. 9. 95·2 ft

EXERCISE XXXIV, Page 203.

1. B_1=59° 37′, B_2=120° 23′ ; A_1=76° 36′, A_2=15° 50′, a_1=10670°, a_2=2992.
2. B=60° or 120°.
3. 57° 10′. 4. No triangle. 5. B=24° 53′ or 155° 7′, C=134° 26′ or 4° 12′, c=232·5 or 23·84.
6. A=26° 12′, B=118° 48′, b=644·3.

7. $B=74° 36'$ or $105° 24'$; $C=65° 24'$ or $64° 36'$; $c=1332$ or 8322.
8. $B=25° 38'$; $C=97° 54'$, $c=62·37$.
9. $A=90°$, $C=43° 48'$, $c=20·95$.
10. Impossible.
11. $C=50° 53'$, $B=34° 51'$.
12. $B=23° 1'$, $c=81·68$ or $B=156° 59'$, $c=3·89$.

EXERCISE XXXV, Page 208.

1. $B=38° 56'$, $c=31° 4''$.
2. $517·2''$.

REVISION QUESTIONS IX, Page 208.

1. 9·6734.
2. $104° 29'$.
3. $A=56° 28'$; $B=25° 38'$; $C=97° 54'$.
4. $38° 13'$; $21° 47'$.
5. $78° 18'$, $49° 36'$.
6. $102° 1'$.
7. $122° 57'$; $16° 3'$.
8. No.
10. $a=31·9$; $b=56·31$; $C=44° 32'$.

EXERCISE XXXVI, Page 212.

1. 2 yds.
2. 69·21 feet nearly.
3. 1·111 mile.
5. $h \cot \beta \tan \alpha$.
8. 100 ft.
10. $13''$ $55°$.

REVISION QUESTIONS X, Page 214.

1. 31 yds.
2. $\dfrac{h \sin \beta \cos \alpha}{\sin (\beta-\alpha)}$, 367·3 ft.
3. 25 ft.
4. $21° 48'$.
5. $\dfrac{h \sin \beta \cos \alpha}{\sin (\beta-\alpha)}$, $\dfrac{h \cos \alpha \cos \beta}{\sin (\beta-\alpha)}$. 54·2.
6. 574 yds.
7. 49·1 ft.
8. 51°.
10. 8486 ft. nearly.
12. 1366 ft. nearly.
15. 160 ft.
18. 100 ft., 25 ft.
19. $\sqrt{\dfrac{b^2(a+b-2h)-h^2(a-b)}{a-b}}$; distance of eye from the cliff is $b\sqrt{\dfrac{a+b-2h}{a-b}}$.

141·7 ft. nearly.
20. $10\sqrt{115}$ ft.
23. 2·02 ft.
24. 200 ft.

EXERCISE XXXVII, Page 227.

1. $14\sqrt{3}$ sq. ft.
2. $12\sqrt{5}$ sq. ft.
3. $7{\cdot}71$ sq. in.
4. 42,50,72.
5. $112°$, $3\frac{3}{8}$ ft.

34. $\dfrac{2lra}{a^2+r^2}$, $\dfrac{(a^2-r^2)b+2ral}{a^2+r^2}$.

REVISION QUESTIONS XI, Page 230.

14. $\dfrac{c \cos A \cos B}{\sin C}$.
15. $\dfrac{c}{2} \cot C$.

16. $\dfrac{1}{3} c \sin B$, $\dfrac{1}{3} a \sin C$, $\dfrac{1}{3} b \sin A$.

EXERCISE XXXVIII, Page 237.

13. $858{\cdot}8$.

MISCELLANEOUS EXERCISES III, Page 247.

4. $(2n+1)\dfrac{\pi}{4}$, $n\pi$.
6. $\dfrac{8}{17}$, $\dfrac{15}{17}$.
7. $30°$ $60°$.

8. $\dfrac{a}{\sqrt{(2+2\sqrt{5})}}$.
9. $\dfrac{n\pi}{2}$.

10. $(2n+1)\dfrac{\pi}{2}$ or $\dfrac{1}{3}\left(2n\pi\pm\dfrac{2\pi}{3}\right)$.

19. $A=56°\ 34'\ 15''$, $C=85°\ 25'\ 45''$; or $A=123°\ 25'\ 45''$, $C=18°\ 34'\ 15''$.

20. 144 ft.
21. $102°\ 1'\ 28''$.

23. $120°$.
24. $\dfrac{a \tan\alpha \tan\beta}{\sqrt{(\tan^2\alpha-\tan^2\beta)}}$.

25. 160 ft.
26. $\dfrac{(m\pm n)^2}{m^2-n^2}$.

29. $\theta+\dfrac{\pi}{4}=2n\pi\pm\left(\alpha+\dfrac{\pi}{4}\right)$.

30. $(4n+1)\dfrac{\pi}{8}$.
37. 105 ft.
57. $60°$.

59. $n\dfrac{\pi}{2}-\dfrac{\pi}{8}$.
64. $\cdot 98$.

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	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7845	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8115	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377		1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	3	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	6	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	6	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences 1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	·0175	89	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	·0349	88	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	·0523	87	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	·0698	86	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	·0872	85	3	6	9	12	15
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	·1045	84	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	·1219	83	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	·1392	82	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	·1564	81	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	·1736	80	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	·1908	79	3	6	9	12	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	·2079	78	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	·2250	77	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	·2419	76	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	·2588	75	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	·2756	74	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	·2924	73	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	·3090	72	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	·3256	71	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	·3420	70	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	·3584	69	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	·3746	68	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	·3907	67	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	·4067	66	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	·4226	65	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	·4384	64	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	·4540	63	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	·4695	62	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	·4848	61	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	·5000	60	3	5	8	10	13
30	·5000	5015	5030	5046	5060	5075	5090	5105	5120	5135	·5150	59	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	·5299	58	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	·5446	57	2	5	7	10	12
33	·5446	5461	5476	5490	5005	5519	5534	5548	5563	5577	·5592	56	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	·5736	55	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	·5878	54	2	5	7	10	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	·6018	53	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	·6157	52	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	·6293	51	2	5	7	9	12
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	·6428	50	2	4	7	9	12
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	·6561	49	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	·6691	48	2	4	7	9	11
42	·6991	6704	6717	6730	6743	6756	6769	6782	6794	6807	·6820	47	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	·6947	46	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	·7071	45	2	4	6	8	10
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

NATURAL COSINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees.	Mean Differences 1′	2′	3′	4′	5′
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	·7193	44	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	·7314	43	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	·7431	42	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	·7547	41	2	4	6	8	10
49	·7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	·7660	40	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	·7771	39	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	·7880	38	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	·7986	37	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	·8090	36	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	·8192	35	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	·8290	34	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	·8387	33	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	·8480	32	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	·8572	31	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	·8660	30	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	·8746	29	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	·8829	28	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	·8910	27	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	·8988	26	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	·9063	25	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	·9135	24	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	·9205	23	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	·9272	22	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	·9336	21	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	·9397	20	1	2	3	4	5
70	·9397	9403	9409	9416	9421	9426	9432	9438	9444	9449	·9455	19	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	·9511	18	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	·9563	17	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	·9613	16	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	·9659	15	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	·9703	14	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	·9744	13	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	·9781	12	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	·9816	11	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	·9848	10	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	·9877	9	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	·9903	8	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	·9925	7	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	·9945	6	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	·9962	5	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	·9976	4	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	·9986	3	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	·9994	2	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	·9998	1	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0	0
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

NATURAL COSINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences 1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	·0175	89	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	·0349	88	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	·0524	87	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	·0699	86	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	·0875	85	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	·1051	84	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	·1228	83	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	·1405	82	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	·1584	81	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	·1763	80	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	·1944	79	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	·2126	78	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	·2309	77	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	·2493	76	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	·2679	75	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	·2867	74	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	·3057	73	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	·3249	72	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	·3443	71	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	·3640	70	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	·3839	69	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	·4040	68	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	·4245	67	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	·4452	66	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	·4663	65	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	·4877	64	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	·5095	63	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	·5317	62	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	·5543	61	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	·5774	60	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	·6009	59	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	·6249	58	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	·6494	57	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	·6745	56	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	·7002	55	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	·7265	54	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	·7536	53	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	·7813	52	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	·8098	51	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	·8391	50	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	·8693	49	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	·9004	48	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	·9325	47	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	·9657	46	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	·0000	45	6	11	17	23	29
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

NATURAL COTANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences				
													1′	2′	3′	4′	5′
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	44	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	43	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	42	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	41	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	1918	40	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	39	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	38	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	37	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	36	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	4281	35	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	34	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	33	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	32	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	31	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	7321	30	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	8040	29	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	8807	28	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	9626	27	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	2·0057	2·0145	2·0233	2·0323	2·0413	2·0503	26	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	·1251	1348	1445	25	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	2460	24	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	3559	23	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	4751	22	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	·6051	21	22	43	65	87	108
69	2·6051	6187	6326	6464	6605	6746	6889	7034	7179	7326	·7475	20	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	·9042	19	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	3·0061	3·0237	3·0415	3·0595	3·0777	18	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	·2709	17	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	·4874	16	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	·7321	15	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	4·0108	14	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	·3315	13	53	107	160	213	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	·7046	12					
78	4·7046	7453	7867	8288	8716	9152	9594	5·0045	5·0504	5·0970	5·1446	11					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	·6713	10					
80	5·6713	7297	7894	8502	9124	9758	6·0405	6·1066	6·1742	6·2432	6·3138	9					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	7·0264	7·1154	8					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	8·0285	8·1443	7					
83	8·1443	2636	3863	5126	6427	7769	9152	9·0579	9·2052	9·3572	9·5144	6					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20	11·43	5					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95	14·30	4					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46	19·08	3					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27	28·64	2					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08	57·29	1					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0	∞	0					
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

Mean differences cease to be sufficiently accurate.

The cotangent of a small angle of *n* minutes of arc or the tangent of 90 minus *n* is very nearly equal to 3438 divided by *n*.

NATURAL COTANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences 1′	2′	3′	4′	5′
0	—∞	7·2419	7·5429	7190	8439	9408	8·0200	8·0870	8·1450	8·1961	8·2419	9					
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206	5428	88					
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041	7188	87					
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	8436	86	21	41	62	82	103
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	9403	85	16	32	48	64	80
5	8·9403	9489	9578	9655	9736	9816	9894	9970	9·0046	9·0120	9·0192	84	13	26	39	52	65
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	0859	83	11	22	33	44	55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	1436	82	10	19	29	38	48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	1943	81	8	17	25	34	42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	2397	80	8	15	23	30	38
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	2806	79	7	14	20	27	34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	3179	78	6	12	19	25	31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	3521	77	6	11	17	23	28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	3837	76	5	11	16	21	26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	4130	75	5	10	15	20	24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	4403	74	5	9	14	18	23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4659	73	4	9	13	17	21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4900	72	4	8	12	16	20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	5126	71	4	8	11	15	19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	5341	70	4	7	11	14	18
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	5543	69	3	7	10	14	17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	5736	68	3	6	10	13	16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	5919	67	3	6	9	12	15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	6093	66	3	6	9	12	15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	6259	65	3	6	8	11	14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	6418	64	3	5	8	11	13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	6570	63	3	5	8	10	13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	6716	62	2	5	7	10	12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	6856	61	2	5	7	9	12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	6990	60	2	4	7	9	11
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	7118	59	2	4	6	9	11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	7242	58	2	4	6	8	10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	7361	57	2	4	6	8	10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	7476	56	2	4	6	8	10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	7586	55	2	4	6	7	9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	7692	54	2	4	5	7	9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	7795	53	2	3	5	7	9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	7893	52	2	3	5	7	8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	7989	51	2	3	5	6	8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	8081	50	2	3	5	6	8
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	8169	49	1	3	4	6	7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	8255	48	1	3	4	6	7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	8338	47	1	3	4	6	7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	8418	46	1	3	4	5	7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	8495	45	1	3	4	5	6
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COSINES

LOGARITHMIC SINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences 1′	2′	3′	4′	5′
45	9·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	8569	44	1	2	4	5	6
46	9·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	8641	43	1	2	4	5	6
47	9·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	8711	42	1	2	3	5	6
48	9·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	8778	41	1	2	3	4	6
49	9·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	8843	40	1	2	3	4	5
50	9·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	8905	39	1	2	3	4	5
51	9·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	8965	38	1	2	3	4	5
52	9·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	9023	37	1	2	3	4	5
53	9·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	9080	36	1	2	3	4	5
54	9·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	9134	35	1	2	3	4	5
55	9·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	9186	34	1	2	3	3	4
56	9·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	9236	33	1	2	3	3	4
57	9·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	9284	32	1	2	2	3	4
58	9·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	9331	31	1	2	2	3	4
59	9·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	9375	30	1	1	2	3	4
60	9·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	9418	29	1	1	2	3	4
61	9·9418	9422	9427	9331	9435	9439	9443	9447	9451	9455	9459	28	1	1	2	3	3
62	9·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	9499	27	1	1	2	3	3
63	9·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	9537	26	1	1	2	3	3
64	9·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	9573	25	1	1	2	2	3
65	9·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	9607	24	1	1	2	2	3
66	9·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	9640	23	1	1	2	2	3
67	9·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	9672	22	1	1	2	2	3
68	9·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	9702	21	0	1	1	2	2
69	9·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	9730	20	0	1	1	2	2
70	9·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	9757	19	0	1	1	2	2
71	9·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	9782	18	0	1	1	2	2
72	9·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	9806	17	0	1	1	2	2
73	9·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	9828	16	0	1	1	2	2
74	9·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	9849	15	0	1	1	1	2
75	9·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	9869	14	0	1	1	1	2
76	9·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	9887	13	0	1	1	1	2
77	9·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	9904	12	0	1	1	1	1
78	9·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	9919	11	0	1	1	1	1
79	9·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	9934	10	0	0	1	1	1
80	9·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	9946	9	0	0	1	1	1
81	9·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	9958	8	0	0	1	1	1
82	9·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	9968	7	0	0	1	1	1
83	9·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	9976	6	0	0	0	1	1
84	9·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	9983	5	0	0	0	0	1
85	9·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	9989	4	0	0	0	0	0
86	9·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	9994	3	0	0	0	0	0
87	9·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	9997	2	0	0	0	0	0
88	9·9097	9998	9998	9998	9998	9999	9999	9999	9999	9999	9999	1	0	0	0	0	0
89	9·9999	9999	10 000	0000	0000	0000	0000	0000	0000	0000	0000	0					
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COSINES

LOGARITHMIC TANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences 1′	2′	3′	4′	5′
0	—∞	7·2419	7·5429	7·7190	7·8439	7·9409	8·0200	8·0870	8·1450	8·1869	8·2419	89					
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208	5431	88					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	7194	87					
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8335	8446	86					
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	9420	85	16	32	48	64	81
5	8·9420	9509	9591	9674	9756	9836	9915	9992	9·0068	9·0143	9·0216	84	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	0891	83	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	1478	82	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	1997	81	9	17	26	35	43
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	2463	80	8	16	23	31	39
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	2887	79	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	3275	78	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	3634	77	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	3968	76	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	4281	75	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	4575	74	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	4853	73	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	5118	72	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	5370	71	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	5611	70	4	8	12	16	20
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	5842	69	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	6064	68	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6263	6257	6279	67	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	6486	66	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	6687	65	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	6882	64	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	7072	63	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	7257	62	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	7438	61	3	6	9	12	15
29	9·7438	7455	7473	7491	7499	7526	7544	7562	7579	7597	7614	60	3	6	9	12	15
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	7788	59	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	7958	58	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	8125	57	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	8290	56	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	8452	55	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	8613	54	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	8771	53	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	8928	52	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	9084	51	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	9238	50	3	5	8	10	13
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	9392	49	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	4999	9514	9529	9544	48	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	9697	47	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	9848	46	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	1000	45	3	5	8	10	13
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COTANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences				
													1′	2′	3′	4′	5′
45	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	0152	44	3	5	8	10	13
46	10·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	0303	43	3	5	8	10	13
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	0456	42	3	5	8	10	13
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	0608	41	3	5	8	10	13
49	10·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	0762	40	3	5	8	10	13
50	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	0916	39	3	5	8	10	13
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	1072	38	3	5	8	10	13
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	1229	37	3	5	8	10	13
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	1387	36	3	5	8	11	13
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	1548	35	3	5	8	11	13
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	1710	34	3	5	8	11	14
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	1875	33	3	5	8	11	14
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	2042	32	3	6	8	11	14
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	2212	31	3	6	9	11	14
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	2386	30	3	6	9	12	14
60	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	2562	29	3	6	9	12	15
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	2743	28	3	6	9	12	15
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	2928	27	3	6	9	12	16
63	10·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3118	26	3	6	9	13	16
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3313	25	3	6	10	13	16
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3514	24	3	7	10	13	17
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3721	23	3	7	10	14	17
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	3936	22	4	7	11	14	18
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4158	21	4	7	11	15	19
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4389	20	4	8	12	15	19
70	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4630	19	4	8	12	16	20
71	10·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4882	18	4	8	13	17	21
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	5147	17	4	9	13	18	22
73	10·514[illegible]	[illegible]	5201	5229	5256	5284	5312	5340	5368	5397	5425	16	5	9	14	19	23
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5719	15	5	10	15	20	25
75	10·571[illegible]	5750	5780	5811	5842	5873	5905	5936	5968	6000	6032	14	5	10	16	21	26
76	10·60[illegible]	6065	6097	6130	6163	6196	6230	6264	6298	6332	6366	13	6	11	17	22	28
77	10·6[illegible]	6401	6436	6471	6507	6542	6578	6615	6651	6688	6725	12	6	12	18	24	30
78	10·6725	6763	6800	6838	6877	6916	6954	6994	7033	7073	7113	11	6	13	19	26	32
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7537	10	7	14	21	28	35
80	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8003	9	8	16	23	31	39
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	8522	8	9	17	26	35	43
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	9109	7	10	20	29	39	49
83	10·9109	9173	9236	9301	9367	9433	9501	9570	9640	9711	9784	6	11	22	34	45	56
84	10·9784	9857	9932	11·0008	11·0085	11·0164	11·0244	11·0326	11·0409	11·0494	11·0580	5	13	26	40	53	66
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	1554	4	16	32	48	64	81
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	2806	3					
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	4569	2					
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167	7581	1	Differences				
89	11·7581	8038	8550	9130	9800	12·0591	12·1561	12·2810	12·4571	12·7581	+∞	0	vary rapidly.				
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COTANGENTS

Borrower's No.	Issue Date	Borrower's No.	Issue Date

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